

# Ebullition in foliated surfaces versus gravitational clumping

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*“Beweisen heißt, den Gedankengang auf den Kopf stellen.”*  
Oswald Teichmüller, 1939, in *Extremale quasikonforme Abbildungen und quadratische Differentiale*.

**ABSTRACT.** For surfaces, we brush a reasonably sharp picture of the influence of the fundamental group upon the complexity of foliated-dynamics. A metaphor emerges with phase-changes through the solid-liquid-gaseous states. Groups of ranks  $0 \leq r \leq 1$  are frozen with intransitivity reigning ubiquitously. When  $2 \leq r \leq 3$ , the marmalade starts its ebullition in the liquid phase, with both regimes (intransitive or not) intermingled after the detailed topology. Whenever  $r \geq 4$ , we reach the gaseous-volatile phase, where any finitely-connected metric surface is transitively foliated. The game extends non-metrically, as putting to the fridge a frozen configuration keeps it frozen. Gromov asked: *Is there a life without a metric?*, yes surely but maybe only a cold eternal one is worth living. The picture is pondered by a scenario of gravitational collapse at the microscopic scale (due to Baillif) dictating the foliated morphology when it comes to surfaces of the Prüfer type (like those of R. L. Moore or Calabi-Rosenlicht).

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Overview and methods . . . . .	5
1.2	Questions and ramifications . . . . .	7
<b>2</b>	<b>Topological preparations</b>	<b>8</b>
2.1	Foundations (Leibniz, Euler, Gauss, Listing, Möbius, Riemann, Klein, Dyck, Schoenflies, Kerékjártó, Radó) . . . . .	8
2.2	Other gadgets: freeness of $\pi_1$ (Ahlfors-Sario) and Whitehead’s spine . . . . .	10
2.3	Indicatrix and orientability (Gauss, Listing, Möbius, Klein, Schläfli, etc.) . . . . .	10
2.4	Dichotomy (Leibniz, Kästner, Bolzano, Jordan, Veblen) . . . . .	12
2.5	Riemann’s branched coverings . . . . .	13
2.6	Dichotomic coverings (via branched coverings) . . . . .	13
2.7	Calibrating the fundamental group (Cannon) . . . . .	14
2.8	Puncturing and cross-capping (Cro-Magnon, von Dyck) . . . . .	15
2.9	Deleting a closed long ray (indicatrix and $\pi_1$ -invariance) . . . . .	16
2.10	Finitely-connected surfaces, cylinder ends (Kerékjártó) . . . . .	16
2.11	The soul of a non-metric finitely-connected surface (Kerékjártó, Nyikos) . . . . .	18
<b>3</b>	<b>Algebraic distractions (Freiheitssätze)</b>	<b>20</b>
3.1	Freeness of the fundamental group of open surfaces (Ahlfors-Sario) . . . . .	20
3.2	Freedom for non-Hausdorff 1-manifolds by reduction to surfaces . . . . .	21
3.3	Twistor or train-tracks (Haefliger, Thurston, Penner) . . . . .	22
<b>4</b>	<b>Foliated foundations</b>	<b>23</b>
4.1	Orienting double cover (Haefliger, Hector-Hirsch, etc.) . . . . .	23
4.2	Compatible flows (Kerékjártó, Whitney) . . . . .	23
4.3	Beck’s technique (plasticity of flows) . . . . .	24
4.4	Foliated triangulations (H. Kneser) . . . . .	24
4.5	Open metric surfaces fibrates (Morse, Thom, etc.) . . . . .	25
<b>5</b>	<b>Haefliger-Reeb theory for non-metric simply-connected surfaces</b>	<b>27</b>
5.1	Haefliger-Reeb derives from Schoenflies (Gauld) . . . . .	27
5.2	Polarized covering à la Riemann and Jordan separation in the large . . . . .	29
5.3	More analogies and divergences from Haefliger-Reeb . . . . .	31
5.4	Hausdorffness of the leaf-space in the $\omega$ -bounded case . . . . .	31
5.5	Missing Euler obstruction . . . . .	32

<b>6</b>	<b>Poincaré-Bendixson arguments</b>	<b>32</b>
6.1	Dynamics on the bottle (Kneser, Peixoto, Markley, Aranson, Gutiérrez)	33
6.2	Dichotomy obstructs oriented transitivity . . . . .	34
6.3	Foliated surfaces with infinite cyclic group . . . . .	34
6.4	Free groups of rank two under dichotomy . . . . .	35
6.5	Free groups of rank two (non-orientable cases) . . . . .	35
6.6	The monolith of finitely-connected metric surfaces . . . . .	37
6.7	Transitive examples via surgery (Peixoto, Blohin) . . . . .	38
6.8	Sporadic obstruction in rank 3 . . . . .	38
6.9	Intransitivity transfer from the metric soul . . . . .	39
6.10	Biminimal foliations (bidirectional denseness) . . . . .	39
<b>7</b>	<b>Gravitational effects (quantum radiation at the microscopic scale)</b>	<b>40</b>
7.1	Long semi-leaves are tame . . . . .	40
7.2	Transitivity implies separability . . . . .	41
7.3	Miniature black holes (Prüfer, R. L. Moore, Baillif) . . . . .	42
7.4	Intransitivity of the thrice-punctured Moore surface . . . . .	43
7.5	Experimental data: prescribing topology and foliated dynamics . . . .	44
7.6	Razor principle foiled . . . . .	47
7.7	Razor principle for long hairs, supernovas and science fiction . . . . .	48
<b>8</b>	<b>Miscellaneous</b>	<b>48</b>
8.1	Jordan separation . . . . .	48
8.2	Grötzsch-Teichmüller theory for non-metric surfaces? . . . . .	49

## 1 Introduction

First the absence of non-metrical nomenclature in our title is somewhat misleading and intended not to discourage potential readers cultivating a broader perspective. Some modest working experience seems to indicate that non-metric manifolds are not studied in autarchy from the metric ones, but rather as an ex-crescence of them. Several truths transcend the metrical barrier with more ease than our psychological apprehension. Much of this plasticity originates from the metrical impulse (*big bang*), and proving a universal statement oft requires working out its metric version first (reminding somehow what Cherry calls the vertical structure of mathematics). By the way the jargon “non-metric manifolds” (like non-Hausdorff manifolds, etc.) is often futile (at least cumbersome), whenever causing an artificial subdivision of statements holding true in a unified setting. Trying to interpret some (existential?) incantation (of D. Hilbert) “*Wir müssen wissen. Wir werden wissen.*” the ultimate verdict would probably involve an infiltration of non-metric manifolds into the real world (assuming its existence, of course) via some geometric modelling (like perhaps, quantum gravity of strings or cytoplasmic vibrations of living beings). Needless-to-say we have no serious idea on how to work this out concretely (yet see Section 7 for some toy examples). At any rate mathematically, it may look puzzling that the continuum and the non-denumerable are well tolerated in the small, yet not very popular in the large.

This is surely a too severe caricature, as non-metric manifolds enjoy a respectable theory (if not a drastic *renouveau* *des matériaux* in R. Thom’s prose) originating with the seminal discoveries of:

- Cantor 1883<sup>1</sup>, Hausdorff 1915, Vietoris 1921, Alexandroff 1924: *long ray* and *long line* (the first found non-metric manifolds, yet maybe not the simplest to visualize),
- Prüfer–Radó 1922–1925, R. L. Moore 1930–1942, Calabi-Rosenlicht 1953: construction of perfectly geometric non-metric surfaces, whose prototype is the so-called *Prüfer surface* discovered near the end of 1922,
- H. Kneser and son M. Kneser: classification of Hausdorff 1-manifolds into 4 species (or 7 in the bordered case) (1958), real-analytic structures on the long 1-manifolds (1960), and a 3-manifold foliated by a *unique* 2D-leaf (1960, 1962),

<sup>1</sup>For references regarding the following chain of items, see the bibliographies of [4], [14].

- M. E. Rudin 1974, Zenor 1975: first set-theoretical independence result involving the concept of *perfect normality* (question of Alexandroff-Wilder),
- Nyikos: bagpipe structures 1984, smoothings of 1-manifolds 1989, long cytoplasmic expansions of surfaces hybridizing Cantor and Prüfer 1990,
- Cannon 1969 [9]: extension of Jordan and Schoenflies to non-metric surfaces and an almost empty, yet not completely nihilist, study of quasi-conformal structures à la Grötzsch 1928-Lavrentieff 1929-Ahlfors 1935-Teichmüller 1938,
- Gauld: independence result for powers, 125 equivalent criteria for the metrizable of a manifold, phagocytosis principle à la Morton Brown: any countable subset of a manifold is contained in a cell ( $\approx$  chart  $\approx \mathbb{R}^n$ ),
- Baillif: homotopical aspects,

More modest recent contributions includes:

- Baillif-Gabard-Gauld 2008 [4]: foliated rigidity in some long manifolds with a cylindrical structure “squat  $\times$  long ray”,
- Gabard-Gauld 2010 [13]: re-exposition of the Jordan-Schoenflies aspects of Cannon 1969,
- Gabard-Gauld 2011 [14]: elementary study of dynamical flows on surfaces mostly, yet with many loose ends.

The present paper is essentially a foliated-dual to the latter article [14]. Whereas in the flow-case *dichotomy* (every Jordan curve separates) is—since Poincaré-Bendixson—a clear-cut barrier to *transitivity* (dense trajectory), the foliated case presents a more subtle landscape modulated by the “size” of the fundamental group and its obstructive influence upon foliated-transitivity (existence of a dense leaf). Precisely when we navigate at low temperatures, say  $\pi_1$  of low-ranks ( $0 \leq r \leq 3$ ), then when  $0 \leq r \leq 1$  the situation is completely frozen (intransitive). As the rank increases to 2 or even 3 the marmalade starts its ebullition in the liquid phase (with pockets of intransitivity still resisting, yet under progressively rarefying circumstances controlled by the topology, cf. Figure 9). Finally as the rank reaches values  $\geq 4$  then we live in the volatile-gaseous regime, where *any metric surface is transitive*. Non-metric extensions take the following form: *frozen-intransitive configurations remain frozen* when imbedded into the cosmic freezer of non-metric manifolds, whereas of course the reverse engineering foils, as putting something liquid or gaseous in the non-metrical fridge may well create a frozen lollypop. This happens for instance to the long plane  $\mathbb{L}^2$ , which punctured as often as you please, still remains intransitive, e.g. by the foliated rigidity previously mentioned [4].

The above metaphoric trichotomy (3 phases delineated by the rank  $r$  of  $\pi_1$ ) quantifies somehow the well-known principle that simple topology impedes complicated dynamics both for *flows* (continuous  $\mathbb{R}$ -actions) as for *foliations* (geometric structures microscopically modelled after the slicing of a number-space  $\mathbb{R}^n$  into parallel  $p$ -planes). Of course the range of the principle is primarily two-dimensional. For instance  $S^3 \times S^3$  admits a *minimal* (=all orbits dense) smooth flow (probably rather chaotic) furnished by a non-constructive Baire type argument of Fathi-Herman (1977). In an earlier paper [14, p. 5], we advanced the naive speculation that positive curvature obstructs the presence of a minimal flow on a closed manifold. If true, the impact is rather gigantic: first all spheres lack minimal flows (Gottschalk conjecture of 1958, still open) and  $S^3 \times S^3$  lack positive curvature (a still older question of Heinz Hopf from the 1930’s).

Back to the more down-to-earth two-dimensionality, the prototype for the above principle is the Poincaré-Bendixson theory, primarily based on the Jordan separation theorem. The latter holds not merely in the plane  $\mathbb{R}^2$ , but in any planar (schlichtartig) surface. In fact Jordan separation holds true non-metrically in simply-connected surfaces (see Gabard-Gauld 2010 [13], and also R. J. Cannon 1969 [9]). To reach the ultimate generality one can adopt Jordan separation as an “axiom” specifying the class of *dichotomic* surfaces and derive the following (via the classical Poincaré-Bendixson trapping-bag argument):

**Lemma 1.1** *A dichotomic surface is flow-intransitive (no dense orbit).*

This applies for instance to the *doubled Prüfer surface*<sup>2</sup>  $2P$ , considered in Calabi-Rosenlicht 1953 [8] (cf. also Figure 2 for an intuitive picture and [14, 5.5] for a proof of  $2P$ 's dichotomy). The same intransitivity as (1.1) is faulty when it comes to foliations. Indeed a noteworthy example of Dubois-Violette 1949 [10, p. 897, Point 4.], smoothly rediscovered in Franks 1976 [11], or Rosenberg 1983 [38, p. 29, V. Rem. 2)], foliates the thrice-punctured plane  $\mathbb{R}_{3*}^2$  by dense leaves. This is manufactured from a foliated disc with two thorns singularities glued with a replica after an irrational rotation (Figure 1). Alternatively it can be regarded as the quotient of Kronecker's irrational winding of the torus divided by the (hyper)-elliptic involution.

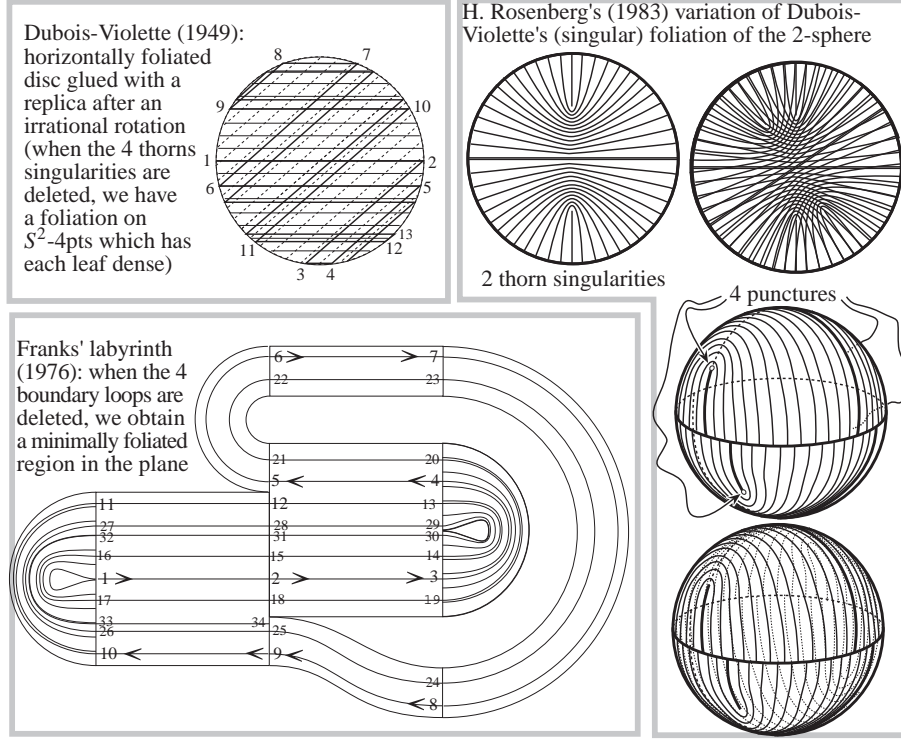


Figure 1: Some labyrinths in quadruply-connected domains of the plane

As recently observed by D. Gauld (in BGG2 [5, Prop.3.1]), the stronger simple-connectivity impedes a dense leaf. Indeed a localized perturbation of the foliation (akin to the closing lemma paradigm) creates a closed leaf (cf. Figure 5, Case 1), which bounds a disc by Schoenflies (an absurdity as it is foliated). Here we use the universal (non-metric) version of Schoenflies presented in [13] (also implicit in Cannon 1969 [9]). In the light of this remark of Gauld, our immediate motivation was two-fold:

- (1) adapt the Haefliger-Reeb theory (1957 [18]) describing foliated structures on the plane  $\mathbb{R}^2$  to any simply-connected (non-metric) surfaces by a purely formal repetition of their arguments,
- (2) exploit this general theory to deduce a somewhat rather special result, saying that *surfaces whose (fundamental) groups  $\pi_1$  are infinite cyclic  $\mathbb{Z}$  also*

<sup>2</sup> In our opinion, the best thing to do is (following e.g., Nyikos 1984 [35]) to define first the *bordered Prüfer surface*  $P$  through a purely geometric process (e.g. like in [12] and the references therein esp. R.L. Moore (1942), and Bredon's book), and then deduce various versions via the operation of collaring, doubling or folding; yielding resp. the classical Prüfer surface  $P_{\text{collar}}$  (appearing first in print in Radó 1925 [37], yet discovered accidentally near the end of 1922 by H. Prüfer), the Calabi-Rosenlicht surface  $2P$  (1953 [8]), and the Moore surface  $M = P_{\text{folded}}$  (1942 in print, yet discovered earlier  $\approx 1930$ , cf. the historiography in [14]). As pointed out by Daniel Asimov, the drawback of our terminology is that it boosts Prüfer's credit vs. Calabi-Rosenlicht contribution. Yet we feel that the bordered viewpoint is very convenient, reducing to a single (geometric) process the generating mode of all those manifolds. Maybe Prüfer's short life justifies anyway some little distortion.

*lack transitive foliations.* This should have involved the universal cover, yet a more Poincaré-Bendixson like method turned out to be more efficient.

Since the torus or punctured torus (with  $\pi_1 = \mathbb{Z}^2$ , resp.  $F_2$  free of rank 2) admit *minimal* foliations (all leaves dense), this exhibits  $\mathbb{Z}$  as the largest possible (fundamental) group impeding a transitive foliation.

During the process of aping non-metrically Haefliger-Reeb (especially the issue that a leaf in a foliated plane divides) we encountered a separation theorem generalizing the separation by a Jordan curve (embedded circle). Specifically any hypersurface which is closed as a point-set (a *divisor* for short) in a simply-connected manifold (of arbitrary dimension) separates the manifold (5.7). This can be deduced from a trick à la Riemann attaching to any divisor  $H$  in a manifold a double (unramified) cover polarized along  $H$ . Intuitively, this covering consists of electrically charged particles, switching their charge signs whenever they cross the hypersurface. (*Warning:* This does not reproves the classical Jordan curve theorem as our hypersurfaces verify a local flatness condition, not a priori known for “wild” Jordan curves in the plane, but true a posteriori via Schoenflies.)

Then questions enchain quite naturally leading to a slightly broader perspective which we shall now try to review. Of course all results must be fairly classical in the metric case (albeit as yet we were not very assiduous in locating references). Indeed even at the metric level our exposition contains some lacunae (maybe the most acute one being our inaptitude to check the foliated-intransitivity of the twice-punctured Klein bottle!), and we hope to manufacture a sharper version in the near future (after some editorial duties).

## 1.1 Overview and methods

Methodically, we can distinguish two trends relying either on the Schoenflies bounding disk property or on the weaker Jordan separation. From the qualitative viewpoint, the former forbids any recurrence to a foliated chart whereas the second permits only moderate recurrences for *oriented* foliations. Coupled with the double cover induced by a non-orientable foliation (4.1) this allows in some favorable situations to draw general conclusions regarding *all* foliations.

The first method (mostly suggested by Gauld) gives the following repetition of Haefliger-Reeb’s results:

(1) *In a (non-metric) simply-connected surface, each leaf appears at most once in a fixed foliated chart* (5.1)<sup>3</sup>. Like in the metric case, this issue is the pillar of a non-metric Haefliger-Reeb theory. Consequently, the leaf-space is a (non-Hausdorff) 1-manifold and leaves are closed as point-sets (but of course open as manifolds). The complete absence of recurrence forbids transitivity. Any leaf is locally flat and thus using Riemann’s covering trick (5.5) it divides the surface (5.1(d)). As a corollary the leaf-space is a simply-connected 1-manifold.

Via the second method based on the weaker Jordan separation, we get the following results using the Poincaré-Bendixson method:

(2) *In a dichotomic surface, an oriented foliation cannot have a dense leaf, nor can a finite union of leaves be dense* (6.4). This follows by examination of the returns of a leaf to a foliated chart, which occur in an orderly fashion. Since foliations of simply-connected manifolds are orientable (4.1), this reproves the intransitivity of such surfaces (without Schoenflies). Besides, *surfaces with infinite cyclic group lack transitive foliations* (6.6). Our proof uses Jordan separation in pseudo-cylinders, namely *orientable surfaces with  $\pi_1 = \mathbb{Z}$  are dichotomic* (2.15). (This fails without orientability as shown by the Möbius band.) When married with Riemann’s trick of branched coverings (familiar in complex function theory, yet pleasant to see at work in the foliated context), the Poincaré-Bendixson method gains more swing. For instance, *dichotomic surfaces with  $\pi_1$  free of rank 2 lack transitive foliations* (6.7). This shows the sharpness of Dubois-Violette’s example: 3 is the minimal number of punctures in the plane to manufacture a transitive foliation (labyrinth). To complete the picture we

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<sup>3</sup>Cross-reference to the main-body of the text.

also notice a non-orientable version: *a non-orientable surfaces with  $\pi_1$  free of rank 2 is foliated-intransitive* (6.10), plus some sporadic obstructions in rank 3 (6.16). (Here is missing the issue with the Klein bottle twice punctured, that we already confessed!)

All these results are first established metrically (with a pivotal reliance on Kerékjártó's cylindrical ends (2.27), as a recipe to compactify metric surfaces of finite-connectivity). The non-metrical boosting involves a Lindelöf exhaustion with calibrated fundamental groups (2.21) (yet another application of Schoenflies) amounting to fill in the holes of a  $\pi_1$ -epimorphic subregion to adjust its group to that of the ambient surface. The logical flattening of the details frequently sidetracked us into purely topological considerations, that were ultimately collected in the first section. Despite the abundance of details, the underlying metabolism remains rather basic:

**Principle 1.2 (Freudo-Lindelöfian Anschauung transfer–FLAT)** *Whatever you are able to see of a non-metric manifold (which is a sort of quantum-plasma in ebullition) it is (in first approximation) its Lindelöf subregions (those truly accessible to the “Anschauung”) which govern both the qualitative “analysis situs” (Jordan, Schoenflies, orientability, dichotomy, fundamental group, etc.) as well as the foliated (or dynamical) destiny of the whole.*

Little corrections are required when a truly non-metric phenomenology is prompted by a particular manifold. Yet this is really a second strata of sophistication not affecting tremendously the generic value of the first principle.

Beside those geometrical methods, we have also a point-set obstruction:

(3) A transitive one-dimensional foliation (abridged 1-foliation) of an  $n$ -manifold  $M^n$  with  $n \geq 2$  implies *separability*<sup>4</sup> of the underlying  $M^n$  (7.3). In fact more is true: any chaotic behaviour of a one-dimensional leaf is caused by one of its metrical short end, whereas long sides of leaves (being sequentially-compact) are always “decently” *properly* embedded (the leaf-topology matches with the relative topology) (7.1).

The above results reflects so-to-speak qualitative features of foliations on some classes of topologically particularized surfaces. A dual aspect is the *quantitative theory*, asking for a classification of foliations (on a fixed manifold). This game, which is almost always hopeless in the metric realm (e.g.,  $\mathbb{R}^2$  is hard yet well-understood,  $S^3$  hopeless), turns out to be much easier on some special non-metric surfaces like, e.g., the long plane  $\mathbb{L}^2$  [4]. Here and on some related surfaces one experiments a rigidity in the large, imposing an asymptotic leaves pattern with freedom left only on certain metric subregions, viz. squares transversally foliated along two opposite sides and tangentially on the remaining two. By the theorem of Kerékjártó-Whitney [27], [43] creating for oriented foliations compatible flows (valid only in the metric case), such a square permits (up to homeomorphism) a unique foliated extension of its boundary data. It followed in [4] that  $\mathbb{L}^2$  tolerates only 2 foliations up to homeomorphism. (This is to be contrasted with the menagerie of foliations grooving the plane  $\mathbb{R}^2$ .)

A plain consequence of this rigidity is the intransitivity of thrice-punctured long-plane  $\mathbb{L}_{3*}^2$  despite its group,  $F_3$  (free of rank 3) (2.23), is one susceptible of complicated foliated dynamics (recall Dubois-Violette). Hence, albeit the fundamental group has much to say, it does not control completely the situation, which depends ultimately upon some finer granularity (encoded in the geometry of the manifold). For an even simpler example, the bagpipe  $\Lambda_{0,4}$  with orientable bag of genus 0 with 4 contours and pipes modelled after the long cylinder  $S^1 \times \mathbb{L}_{\geq 0}$ , is a (dichotomic) surface with  $\pi_1 = F_3$ , yet intransitive. In fact  $\Lambda_{0,4}$  cannot even be foliated [4], because any pipe acts as a black hole aspirating leaves in a purely vertical fashion or creating many horizontal circle leaves  $S^1 \times \{\alpha\}$ . Thus an appropriate surgery reduces one to the compact Euler-Poincaré obstruction. Both examples cited are trivial, inasmuch as their intransitivity also derives from their non-separability (via (3) above).

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<sup>4</sup>Existence of a countable dense subset.

The transitivity decision problem becomes more perfidious if one wonders about the transitivity of the separable,  $M_{3*}$ , thrice-punctured Moore surface. (Recall that the Moore surface is the folded Prüfer surface, cf. Figure 2 for an intuitive picture.) Albeit there is no universal algebraic obstruction (recall again Dubois-Violette), we experiment a geometric one related to the granularity of the Moore surface (7.7). Indeed an argument of M. Baillif (Buenos Aires era, near 2008–2009, under press in [5] and reproduced below (7.6)) the “thorns” of the Moore surface (i.e., the folded images of the boundaries of Prüfer) acts as a “continuum” series of miniature black holes inveigling *almost all* leaves. Precisely, *almost all thorns (all but at most countably many exceptions) are semi-leaves of any foliated Moore surface* (7.6). This implements a scenario of gravitational collapse at the microscopic scale in sharp contrast—but somehow dual—to the macroscopic scale at which lives the (super-massive) black hole sitting at the long end of a Cantor cylinder  $S^1 \times \mathbb{L}_+$  [4]. All these examples imaginatively suggest to contemplate foliated structures (like lignite distributions) merely a mean of evidencing the magneto-gravitational ( $\approx$ geometric, since Newton-Euler vs. Leibniz-Euler(!)-Riemann-Einstein<sup>5</sup>) anomalies of the underlying manifold.

The above results do not tell whether the doubled Prüfer surface  $2P$  accepts a labyrinth (=transitive foliation). Yet, it probably does in view of the toy:

**Example 1.3** (*Dubois-Violette Prüferized*) Taking Dubois-Violette’s foliated disc (Fig. 1, Rosenberg’s version) and Prüferize the 2 leaved-arcs ending to the 2 punctures to get a bordered surface  $\wp$ , which glued with a copy after an irrational twist, yields a separable surface  $2\wp$  transitively foliated, which is non-metric, dichotomic, etc. and very resemblant to  $2P$ .

It is not impossible (and indeed highly probable) that this Prüferized surface  $2\wp$  is homeomorphic to  $2P$  (just “magnify” some bridges). Yet this foliation is not minimal, and it is natural to wonder if  $2P$  is minimally foliated (more about this soon!). For flows, an easy argument of propagation [14] showed that minimality forces metrisability, raising some hope to classify all flow-minimal surfaces. Presumably not so with foliated surfaces, compare [5] for some minimally foliated non-metric surfaces. The simplest example, depicted on Figure 12 below, involves a punctured torus minimally foliated à la Kronecker with one of the two leaves ending to the puncture elongated up to reach the length of Cantor’s long ray). This Pinocchio expansion near the puncture exploits a construction of Nyikos (cf. [5] for more details and the original reference).

Now what about  $2P$  being minimally foliated? Since the fundamental group is extremely voluminous (free on a continuum of generators), the rank is big, pushing the surface in the very gaseous-volatile regime where transitivity mutates in minimality. However the real answer is quick and easy thanks to the gravitational clumping of Baillif, to the effect that a (finally violent) condensation of diffuse gas must occur along the “bridges” (i.e. the images of the boundaries of  $P$  in  $2P$  via the canonical inclusion  $P \hookrightarrow 2P$ ). Indeed arguing as for the Moore surface (7.6), Baillif’s method shows that in any foliation of  $2P$  *almost all* bridges are leaves (as above this means all but countably many exceptions). Thus we have with  $2P$  (or better its Dubois-Violette model  $2\wp$  (1.3)) a surface which is transitive, yet not minimal. (The author does not know if such an example exist metrically.)

## 1.2 Questions and ramifications

Here we mention a short list of questions which are probably not structurally hard, but rather unsolved due to the incompetence of the writer.

(1) *Some metrical missing links.* As just noticed what is the simplest example of a metric surface which is transitively foliated but not minimally. Also is the twice punctures Klein bottle  $\mathbb{K}_{2*}$  foliated-intransitive? This is actually the only

<sup>5</sup>Compare, e.g., the historiography in Speiser 1927 [42].

missing case to complete our picture (Figure 9) classifying finitely-connected surfaces according to their foliated-transitivity. Which metric surfaces can be *biminally foliated* (i.e. so that each semi-leaves are dense)? Cf. (6.21) for a partial answer.

(2) *Any pseudo-Moore surface foliates?* In the case of flows, the *phagocytosis lemma* (saying that any countable subset of a manifold is contained in a chart) found a nice application to what we called in [14] the *pseudo-Moore problem* (no non-singular flow on a non-metric, simply-connected, separable surface). Such surfaces are referred to as *pseudo-Moore*, with the Moore surface being the simplest prototype. In the foliated case, it is not obvious to guess an applied avatar, except for the over-optimistic option that all pseudo-Moore surfaces foliate. Recall that for flows, the Moore surface had no brush, and this turned out to be the fate of any pseudo-Moore surface [14]. But now the Moore surface foliates, thus should we expect that any pseudo-Moore surface foliates? We believe the answer is negative, in view of Nyikos' long cytoplasmic expansions (cf. the discussion following Question (7.9)).

(3) *Euler obstruction in the  $\omega$ -bounded case.* Another frustrating problem is what happens to the Euler-Poincaré obstruction? Specifically we conjecture that  $\omega$ -bounded<sup>6</sup> surfaces with  $\chi < 0$  lack foliations (independently of any specification of the pipes). In the case of flows, it was comparatively easy to show [15] that a non-vanishing  $\chi \neq 0$  obstructs non-singular flows (non-metric hairy-ball theorem).

(4) *Freeness of the fundamental group of curves (=non-Hausdorff 1-manifolds) and the Haefliger twistor.* Can somebody prove the (hypothetical) Lemma 3.4 below, which seems to be folklore since Haefliger 1955 [17], and which could play a crucial rôle in showing that all (non-Hausdorff) 1-manifolds have a free fundamental group. Prior to this we show the  $\pi_1$ -freeness of all open (Hausdorff) surfaces by reduction to the metric case (3.1). (Hausdorffness is of course essential, as seen by picturing flying-saucers, e.g.,  $S^1 \times$  (line with two origins) with  $\pi_1 = \mathbb{Z}^2$ .)

In guise of provisory conclusion, we diagnostic that our understanding of foliated structures (especially on surfaces) is slightly less sharp than the corresponding one for dynamical flows, where deeper paradigms entered effectively into the arena (like phagocytosis or the Euler-Poincaré obstruction).

## 2 Topological preparations

This section collects purely topological results, independent of (yet related to) our foliated investigations. The reader can skip it referring to individual results later if necessary. The Leitfaden below is supposed to help navigating through the menagerie of details.

### 2.1 Foundations (Leibniz, Euler, Gauss, Listing, Möbius, Riemann, Klein, Dyck, Schoenflies, Kerékjártó, Radó)

We first recall without proofs (but cross-references) the key results in the topology of the plane and surfaces. A pillar of the theory is the following theorem often attributed to Schoenflies (1906), albeit there are serious function-theoretical competitors building over the Riemann mapping theorem and conformal representation (including Osgood 1900–1913, Carathéodory 1912, Hilbert-Courant, etc.), not to mention the early attempt in Möbius 1863 [33]:

**Theorem 2.1** (*Schoenflies 1906*) *Any Jordan curve (embedded circle) in the plane  $\mathbb{R}^2$  bounds a disc.*

**Proof.** Compare e.g. Siebenmann 2005 [41]. ■

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<sup>6</sup>A space is  $\omega$ -bounded if countable subsets have compact closures. In the manifold-case, this amounts to Lindelöf subregions having compact closures. This concept is a non-metric avatar of compactness, especially acute for surfaces in view of Nyikos' bagpipe theorem.





**Proof.** It is probably fair to qualify all early proofs as semi-intuitive inasmuch as they required some geometric ‘structuration’ lying beyond the naked topological manifolds (those were perhaps first defined in print in Kerékjártó 1923 [26], though the idea is much older, e.g., Riemann 1. March<sup>7</sup> 1853–1854, Betti, Poincaré 1895, Tietze 1907, Brouwer, Weyl 1913, etc.). Thus one first triangulates the surface with Radó (2.3) and then apply the combinatorial reduction to a normal form à la Dehn-Heegaard, say or alternatively do Möbius-Morse theory. For a modern book form, cf. e.g., Massey 1967 [31]. ■

We now list several consequences, starting with the following, historically perhaps first proved via the uniformization theorem (Klein, Poincaré, Koebe 1882–1907). Recall also an alternative proof via triangulations and the combinatorial device of van der Waerden-Reichardt (ref. as in [13] (arXiv version)):

**Proposition 2.5** *A metric simply-connected surface is either  $S^2$  or the plane.*

**Proof.** Cf. also Ahlfors-Sario [1] and Massey [31] (in the exercise). ■

This in turn implies first the metric-case of the following:

**Lemma 2.6 (Homotopic Schoenflies)** *(Baer 1928, Cannon 1969) A null-homotopic Jordan curve in a surface (metric or not) bounds a disc. In particular any Jordan curve in a simply-connected surface bounds a disc, which is unique whenever the surface is open (equivalently not the sphere).*

**Proof.** Via passage to the universal covering (still metric by Poincaré-Volterra and the countability of the  $\pi_1$  ensured by Radó’s triangulation (2.3), or alternatively just lift the triangulation and use Weyl (2.2)), we may apply in view of (2.5) the classic Schoenflies theorem (2.1). An argument of R. Baer, 1928 (compare e.g., [13]), shows that the bounding disc for the lifted Jordan curve is homeomorphically projected down in the original surface.

The non-metric case reduces to the metric one, by covering the range of a null-homotopy by a Lindelöf subregion (as observed in Cannon 1969 [9]). ■

## 2.2 Other gadgets: freeness of $\pi_1$ (Ahlfors-Sario) and Whitehead’s spine

**Lemma 2.7** *The fundamental group of an open metric surface is free on countably many generators.*

**Proof.** Cf. Ahlfors-Sario 1960 [1, §44A., p. 102] or Massey 1967 [31]. ■

Using Whitehead’s spine we get the stronger assertion:

**Lemma 2.8** *Any open metric surface retracts by deformation onto a subgraph of the 1-skeleton of any of its triangulation. In particular it is homotopy equivalent to a (countable) graph.*

**Proof.** The theory of the spine originates in Whitehead 1939, cf. also Massey’s book 1967 [31] for a discussion. ■

## 2.3 Indicatrix and orientability (Gauss, Listing, Möbius, Klein, Schläfli, etc.)

Those classical notions (originating with the discovery (circa 1860) of the *Möbius band* involving a well-documented( $\pm$ ) question of priority between close colleagues, namely Gauss and Listing) is clearly independent of a metric and makes sense for all manifolds. Several viewpoints are possible (combinatorial vs. naked TOP-manifolds). A first “naked” aspect is to define the *indicatrix* (or even better the *orientation covering*):

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<sup>7</sup>Compare, e.g., Speiser 1927 [42, p. 107-8]

**Lemma 2.9** *Given a manifold  $M$ , one can propagate “local orientations” around loops to obtain a morphism  $\pi_1(M) \rightarrow \{\pm 1\}$  (called the indicatrix). The latter is in fact just the monodromy of  $M_\circlearrowleft \rightarrow M$  the double orientation cover given by de-doubling points by their two possible local orientations. Being purely local, the construction works for all locally Euclidean spaces even without Hausdorff proviso, and being perfectly intrinsic it has the following:*

*(Naturality) If  $L \subset M$  is a subregion of the manifold  $M$ , then its orientation covering  $L_\circlearrowleft \rightarrow L$  is just the restriction of that of  $M$  to  $L$ .*

**Proof.** It boils down to define “local orientations”, cf. e.g. Dold’s Algebraic Topology. ■

A manifold is said to be *orientable* if its indicatrix is trivial (equivalently, if its orientation covering is trivial).

**Lemma 2.10** • *(Hereditiy) Any subregion of an orientable manifold is orientable.*

• *(Transfer) A manifold all of whose Lindelöf subregions are orientable is orientable.*

**Proof.** The hereditary claim reduces to the fact that triviality of a covering is preserved by restricting to subregions of the base. The transfer claim requires a little argument. If not orientable the given manifold,  $M$ , has a non-trivial orientation-covering, which is therefore connected. Thus there is a path in  $M_\circlearrowleft$  connecting the two points lying above the (arbitrarily) fixed basepoint of  $M$ . This path, being compact, is contained in some Lindelöf subregion, which up to taking a Lindelöf exhaustion of the base  $M$  can be assumed to be the inverse image of a Lindelöf subregion  $L$  of  $M$ . By naturality (2.9)  $L$  is non-orientable, violating the assumption. ■

Another common definition of orientability (of a manifold of any dimensionality) is that any embedded circle has a trivial tubular neighbourhood. Yet we probably want to exclude wild knots. Several respectable theories (PL, DIFF, etc.) explain how to tame wildness, yet as we are primarily concerned with the 2D-case there is an intrinsic weapon namely Schoenflies (2.1) and a resulting tubular neighbourhood theory (cf. e.g., Siebenmann 2005 [41]) permitting to circumvent any specialisation to such structures (whose existence is rather weak in the non-metric context and as we know even for compact manifolds not universally available as soon as the dimension is  $\geq 4$ ).

**Lemma 2.11** *A surface is orientable iff any Jordan curve has a trivial tubular neighbourhood. In particular puncturing finitely many points in a surface does not affect the indicatrix.*

**Proof.** [ $\Rightarrow$ ] Let  $J$  be a Jordan curve in the surface  $M$ , and let  $T$  be its tubular neighborhood, which is an  $\mathbb{R}$ -bundle over the circle  $S^1$ . By classical bundle theory there is only two such bundles: the trivial one and a twisted one (the open Möbius band). The latter option is precluded by heredity (2.10).

[ $\Leftarrow$ ] The converse looks more tricky, and we are only able to perform a reduction to the metric case. Let  $L$  be a Lindelöf subregion of  $M$ . Then clearly the assumption of triviality of Jordan neighborhoods holds in  $L$  as well, thus by the metric case of the lemma,  $L$  is orientable. By transfer (2.10) it follows that  $M$  is orientable.

*Metric case (outline).* The proof in the metric case works maybe as follows: fix a triangulation and subdivide barycentrically until all 2-simplexes lie in charts. Then local orientations takes a more down-to-earth interpretation as the borders of those simplices. Now the Jordan triviality assumption specialized to combinatorial loops ensure that there is a coherent way to orient simplices in the combinatorial sense, implying the topological sense of (2.9). (Exercise: find a reference where this is properly done, e.g. Möbius 1865, Weyl 1913, etc.)

For the last clause, just observe that if the original surface is non-orientable, then it contains a Möbius band and the punctures can be performed outside of it. ■

## 2.4 Dichotomy (Leibniz, Kästner, Bolzano, Jordan, Veblen)

After a long series of precursors (and successors), Jordan (1887) showed that any embedded circle in the plane disconnects the plane in two components. This motivates the following jargon (borrowed from O. Hájek [20]):

**Definition 2.12** A surface is *dichotomic* if any Jordan curve (=embedded circle) divides the surface. More common synonyms are *planar* (or *schlichtartig*), yet both sound too restrictive when it comes to allow non-metric surfaces.

Using homology and the five lemma, one can show (cf. [14, 5.3, 5.4]):

**Lemma 2.13** • (Heredity) Any subregion of a dichotomic surface is dichotomic.  
• (Transfer) A surface all of whose Lindelöf subregions are dichotomic is dichotomic.

**Lemma 2.14** *A dichotomic surface is orientable.*

**Proof.** In view of the orientability criterion (2.11), let  $J$  be a Jordan curve in the surface  $M$ , and let  $T$  be a tube around it. By heredity of dichotomy (2.13) the latter is dichotomic, hence cannot be the Möbius band. ■

While the converse of (2.14) is not true (e.g., torus), it is sometimes:

**Lemma 2.15** *An orientable surface with infinite cyclic group is dichotomic.*

**Proof.** (The following argument is homological, so rather algebraic; for a more geometric proof using Schoenflies, see Remark 2.19.) Let  $J$  be a Jordan curve in the surface  $\Sigma$ . We can assume that  $J$  is not null-homotopic, since otherwise Jordan separation is obvious as  $J$  bounds a disc (2.6). We fix  $T$  a tubular neighbourhood of  $J$ , which is trivial, i.e.  $T \approx \mathbb{S}^1 \times \mathbb{R}$  (since  $\Sigma$  is orientable). To show that  $\Sigma - J$  is disconnected we examine the homology exact sequence of the pair  $(\Sigma, \Sigma - J)$  written down as the third line of the diagram below. Just above it we have the sequence of the tube pair  $(T, T - J)$ , which we embed as the complement of the poles of the 2-sphere (denoted  $S$ ) while mapping  $J$  to the equator. This gives us the first line which is the sequence of the pair  $(S, S - J)$ . By naturality all squares are commutative.

[illegible]

The excision isomorphisms are denoted by vertical equivalence symbols. Bold-face “**0**” symbols indicate trivial groups, while other bold indices indicate the rank of the corresponding abelian group. Looking at the first line we find that  $t = 1$  and  $u = 1$ , which values propagates downstairs by the excision isomorphisms. Next the group  $H_2(\Sigma) = 0$  is trivial, because  $\Sigma$  is an open 2-manifold and the postulated fundamental group  $\mathbb{Z}$  does not occur among the list of closed surfaces. (We used implicitly the vanishing of the top-dimensional homology  $H_n$  of Hausdorff  $n$ -manifolds, compare e.g. Samelson 1965 [39].) Thus by exactness of the bottom line, we have  $1 - s + 1 - 1 + r - 1 = 0$ , i.e.  $r = s$ , provided all ranks are finite. For this we apply the *five lemma* saying that if the diagram of abelian groups has exact rows and each square is commutative:

$$\begin{array}{ccccccccc} C_1 & \rightarrow & C_2 & \rightarrow & C_3 & \rightarrow & C_4 & \rightarrow & C_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ D_1 & \rightarrow & D_2 & \rightarrow & D_3 & \rightarrow & D_4 & \rightarrow & D_5 \end{array}$$

Then

(1) if  $f_2$  and  $f_4$  are onto and  $f_5$  injective, then  $f_3$  is onto.

(2) if  $f_2$  and  $f_4$  are injective and  $f_1$  is onto, then  $f_3$  is injective.

Part (1) does not apply to the  $f_i$  (we do not know  $f_4$  to be onto), but it applies to the  $g_i$  showing that  $g_3$  is onto, so  $r \leq 2$ . Since the group indexed by  $s$  is squeezed in an exact sequence with zeros extremities, it has finite rank as well. Now part (2) applies to the  $f_i$  (but not to the  $g_i$ !), thus  $f_3$  is injective and  $s \geq 2$ , so  $r \geq 2$  as we knew  $r = s$ . This completes the proof. ■

## 2.5 Riemann's branched coverings

The following mechanism originating in complex function theory (Riemann's Thesis 1851) will later find a pleasant application to foliated structures:

**Lemma 2.16** *Given a finite  $d$ -sheeted covering  $\Sigma \rightarrow F_{n*}$  of a punctured surface  $F$ , there is a canonical recipe to fill over the punctures to deduce a branched covering  $\Sigma^* \rightarrow F$  whose total space is a surface. Moreover if the (unpunctured) surface  $F$  is compact, then so is  $\Sigma^*$ , and their Euler characteristics are related by the so-called Riemann-Hurwitz formula:*

$$\chi(\Sigma^*) = d\chi(S^2) - \deg(R), \quad (1)$$

where  $\deg(R)$  is the ramification counted with multiplicity. Further orientability of  $F$  transfers to  $\Sigma^*$ .

**Proof.** If we look at a “pierced neighbourhood”  $U$  of a puncture  $p \in F - F_{n*}$  topologically like  $\mathbb{C}^*$  (punctured complex plane) we obtain a covering  $p^{-1}(U) \rightarrow U$ . Since  $\pi_1(U)$  is  $\mathbb{Z}$ , the coverings of  $U$  are completely classified, being the mappings  $z \mapsto z^k$  (from  $\mathbb{C}^*$  to itself) for some integer  $k \geq 1$ . So there is a natural way to fill over the punctures (Riemann's trick) to obtain  $\Sigma^* \rightarrow F$  a branched covering of degree  $d$  whose total space  $\Sigma^*$  is a surface.

The Riemann-Hurwitz formula follows by a Euler characteristic counting. Triangulate  $F$  so that punctures are vertices, and lift simplices to  $\Sigma^*$  and count the alternating sum of those, which behaves multiplicatively up to the correction effected by ramification.

The assertion regarding orientability can be checked combinatorially, or by noticing that puncturing does not affect orientability. Hence  $F$  orientable implies  $F_{n*}$  orientable (2.10), and in turn the covering  $\Sigma$  is orientable, and finally  $\Sigma^*$  is orientable. (Little exercises.) ■

## 2.6 Dichotomic coverings (via branched coverings)

The following specialization of (2.16) will be useful for the sharpness of Dubois-Violette's labyrinths (i.e., 3 is the minimal number of punctures required in the plane to construct a transitive foliation):

**Lemma 2.17** *The total space of a double covering  $p: \Sigma \rightarrow M$  of a dichotomic surface  $M$  with  $\pi_1(M) = F_2$  is itself dichotomic.*

**Remark 2.18** The result is sharp as shown by the standard branched covering  $T^2 \rightarrow S^2$  ramified at 4 points: divide the torus by the (holomorphic) involution  $z \mapsto -z$ , or, rotate by  $180^\circ$  a Euclidean model of the torus in revolution.

**Proof.** We first establish the metric case and then boost the result beyond the metrical barrier via the usual exhaustion method. The metric case involves the trick of branched coverings (2.16).

**Metric case.** By (a special case (2.29) of) Kerékjártó's classification (2.27),  $M$  is homeomorphic to  $S_{3*}^2$  (sphere with 3 punctures), and we compactify  $M$  to the sphere  $S^2$  by adding 3 points. By filling over the punctures (Riemann's trick) we obtain  $\Sigma^* \rightarrow S^2$  a branched covering of degree 2. The space  $\Sigma^*$  is a

surface which is compact, borderless and orientable ( $M$  being dichotomic, hence orientable (2.14), thus so is  $\Sigma^*$ ). By Riemann-Hurwitz we have

$$\chi(\Sigma^*) = 2\chi(S^2) - \deg(R), \quad (2)$$

where  $\deg(R)$  is the ramification counted with multiplicity. Since the degree of the map is 2 there is only simple ramification so that  $\deg(R)$  is just the cardinality of the branched points. In our situation,  $\deg(R) \leq 3$  and since  $\chi(\Sigma^*) = 2 - 2g$  where  $g$  is the genus, we deduce that  $g = 0$ . By classification (2.4)  $\Sigma^*$  is the sphere, which is dichotomic by the Jordan curve theorem. Thus  $\Sigma$  is dichotomic as well by heredity (2.13).

**Non-metric case.** We choose an exhaustion  $M = \bigcup_{\alpha < \omega_1} M_\alpha$  by Lindelöf subregions  $M_\alpha$  and we may arrange  $\pi_1(M_\alpha) \approx F_2$ . Such a “calibration” of the fundamental group is justified in Lemma 2.21 below. Since  $M$  is dichotomic, the  $M_\alpha$  are also dichotomic (2.13). Thus  $\Sigma_\alpha := p^{-1}(M_\alpha) \rightarrow M_\alpha$  is dichotomic as well by the metric case. Now given  $L$  a Lindelöf subregion of  $\Sigma$ , there is some  $\alpha$  such that  $L \subset \Sigma_\alpha$ . By heredity  $L$  is dichotomic, and the transfer (2.13) completes the proof. ■

**Remark 2.19** This argument reproves (2.15), i.e. *dichotomy of orientable surfaces with infinite cyclic group*. Let us carry out this simple exercise.

**Another proof of 2.15.** By (2.21) we have an exhaustion  $M = \bigcup_{\alpha < \omega_1} M_\alpha$  by Lindelöf (hence metric) subregions with  $\pi_1(M_\alpha) \approx \mathbb{Z}$ . Since  $M$  is orientable, so are the  $M_\alpha$  which are therefore open cylinders (again by an appropriate special case of Kerékjártó (2.27)), hence in particular dichotomic. By transfer (2.13) it is enough to show that any Lindelöf subregion  $L$  of  $M$  is dichotomic, and so is the case by heredity (2.13) because  $L$  is contained in some  $M_\alpha$ , which is dichotomic. ■

## 2.7 Calibrating the fundamental group (Cannon)

We now check the pivotal lemma about exhaustions respecting the fundamental group (which is yet another consequence of Schoenflies going back to R. J. Cannon 1969 [9, p. 98], “fill in the holes” argument). First we show a kernel killing procedure. *Warning:* our clumsy(?) proof uses beside Schoenflies, some other gadgets like the freeness of the fundamental group of open metric surfaces (2.7) plus the stronger theory of J.H.C. Whitehead’s spine (2.8) telling that such surfaces retract by deformation to a countable graph—referred to as ‘the’ spine.

**Lemma 2.20** (*Kernel killing procedure*) *Given a Lindelöf subregion  $L$  in a surface  $M$  so that the natural map  $\pi_1(L) \rightarrow \pi_1(M)$  is epimorphic, there is a larger Lindelöf subregion  $L' \supset L$  such that  $\pi_1(L') \rightarrow \pi_1(M)$  is isomorphic.*

**Proof.** If the natural morphism  $j: \pi_1(L) \rightarrow \pi_1(M)$  is not injective, then for any element of the kernel we have a shrinking homotopy whose compact range may be covered by finitely many charts which aggregated to  $L$  gives some  $L^*$ . Since  $L$  is metric, its group  $\pi_1(L)$  is countable, and we need only iterate countably many times the procedure, thereby conserving Lindelöfness for the enlarged  $L^*$ . It may seem that  $\pi_1(L^*) \rightarrow \pi_1(M)$  is now isomorphic. However when killing an element of the kernel may well accidentally create a parasite “handle” or “connectivity”, jeopardizing the desideratum. The trick is to take advantage of some geometric topology à la Schoenflies, to kill (or better plumb) the holes in a surgical way, without generating new ones by inadvertence. Suppose first that an element in  $\ker j$  is represented by a Jordan curve, which being null-homotopic in  $M$  bounds a disc in  $M$  (2.6), which aggregated to  $L$  kill one holes without creating new ones. Unfortunately, not all element of the  $\pi_1$  of a surface are representable by Jordan curves (e.g., the generator squared in the group of a punctured plane is not Jordan-representable). By carefully

selecting who to kill in the kernel, namely the primitive elements (yet not their proper powers), as the former admit Jordan representants (cf. (2.22) below) completes the procedure. As countably many discs are aggregated Lindelöfness is preserved. Also killing the primitive elements of the kernel suffices to kill the whole kernel. Indeed the latter is a subgroup of  $\pi_1(L)$  which is known to be free when  $L$  is an open surface, hence free as well and therefore generated by its primitive elements. Of course assuming  $L$  open is not expensive, since otherwise  $L$  is compact, hence clopen, so identic to  $M$  and the lemma is trivially true. ■

**Lemma 2.21** *A surface  $M$  with finitely (or countably) generated fundamental group has an exhaustion by Lindelöf subregions  $M_\alpha$  such that the morphisms  $\pi_1(M_\alpha) \rightarrow \pi_1(M)$  induced-by-inclusions are isomorphic for all  $\alpha$ .*

**Proof.** Choose a finite (or countable) generating system of  $\pi_1(M)$ , and representing loops  $c_i$ . Each  $c_i: [0, 1] \rightarrow M$  is a continuous map with  $c_i(0) = c_i(1) = \star$  the basepoint of  $M$ . Cover randomly the range of the  $c_i$  by charts to get a Lindelöf subregion  $L_0$ . By construction  $\pi_1(L_0) \rightarrow \pi_1(M)$  is epimorphic, and by kernel killing (2.20) we find  $M_0 := L'_0$  with the required properties of Lindelöfness and incompressibility. Then aggregate randomly countably many new charts to get  $L_1 \supset M_0$  and again kernel killing  $\pi_1(L_1) \rightarrow \pi_1(M)$  gives  $M_1$ . Transfinite induction completes the proof by defining  $M_\lambda$  to be a kernel killing enlargement of  $L_\lambda = \bigcup_{\alpha < \lambda} M_\alpha$  whenever  $\lambda$  is a limit ordinal. ■

**Lemma 2.22** *In the fundamental group of an (open) metric surface  $M$  any primitive element (i.e., not a proper power) is representable by a Jordan curve.*

**Proof.** Our argument is not very intrinsic relying on combinatorial methods. (Is there an argument via the universal covering?) Via Whitehead's spine (2.8),  $M$  retracts by deformation  $M \rightarrow \Gamma$  to a countable graph  $\Gamma$ . By the primitivity assumption, the loop pushed in the spine is homotopic to a simple loop (imagine an edge in the bouquet of circles resulting by collapse of a maximal tree), hence representable by a Jordan curve in the graph, so a fortiori in the surface  $M$ . ■

## 2.8 Puncturing and cross-capping (Cro-Magnon, von Dyck)

This section gives algebraic arguments for two intuitively obvious issues:

**Lemma 2.23** *Puncturing an open surface adds one free generator to the fundamental group.*

**Proof.** If  $S$  is open, any puncture increases by one the rank of the  $H_1$ . This follows e.g. by writing the exact sequences of the pairs  $(S, S_* = S - 1pt)$  and  $(U, U_*)$ , where  $U$  is a chart containing the puncture:

$$\begin{array}{ccccccccccccccc} H_2(S) & \rightarrow & H_2(S, S_*) & \rightarrow & H_1(S_*) & \rightarrow & H_1(S) & \rightarrow & H_1(S, S_*) & \rightarrow & H_0(S_*) & \rightarrow & H_0(S) & \rightarrow & H_0(S, S_*) & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H_2(U) & \rightarrow & H_2(U, U_*) & \rightarrow & H_1(U_*) & \rightarrow & H_1(U) & \rightarrow & H_1(U, U_*) & \rightarrow & H_0(U_*) & \xrightarrow[\cong]{f_1} & H_0(U) & \rightarrow & H_0(U, U_*) & \rightarrow & 0 \end{array}$$

Hence if  $\pi_1(S)$  is free of rank  $r$ , then  $\pi_1(S_*)$  being free by (3.1) is free of rank  $r + 1$ . In particular if  $S$  is 1-connected and open, then  $\pi_1(S - k \text{ pts})$  is  $F_k$ , free of rank  $k$ . ■

Likewise cross-capping has the same impact on the fundamental group:

**Lemma 2.24** *Cross-capping an open surface adds one free generator to the fundamental group.*

**Proof.** Recall the cross-capping operation (von Dyck, 1888) of a surface  $M$  amounting to identify diametrically opposite points at the border of a embedded compact disc  $D \subset M$ . Denote  $M_c$  the cross-capped surface. Choose  $U$  a neighborhood of the cross-cap, which is homeomorphic to an (open) Möbius band. We have  $M_c = U \cup M_\star$ , where  $M_\star = M - D$ . The Mayer-Vietoris sequence:

$$\underbrace{H_2(M_c)}_0 \rightarrow \underbrace{H_1(U \cap M_\star)}_Z \rightarrow H_1(U) \oplus H_1(M_\star) \rightarrow H_1(M_c) \rightarrow H_0(U \cap M_\star) \rightarrow H_0(U) \oplus H_0(M_\star),$$

whose last arrow is injective, truncates as  $H_1(M_c) \rightarrow 0$ . The first group is trivial since  $M_c$  is open. It follows that the rank of  $H_1(M_c)$  equals that of  $H_1(M_\star)$ , which is one more than that of  $H_1(M)$  (puncturing a compact disc amounts to puncture a point and use (2.23)). The claim now follows by the freeness of the  $\pi_1$  (3.1). ■

## 2.9 Deleting a closed long ray (indicatrix and $\pi_1$ -invariance)

The *closed long ray*  $\mathbb{L}_{\geq 0}$  is the unique bordered non-metric (Hausdorff) 1-manifold.

**Lemma 2.25** *Given a closed long ray  $L$  embedded in a surface  $M$ , the surface  $M - L$  slitted along  $L$  has the same  $\pi_1$ . In fact the natural morphism  $\pi_1(M - L) \rightarrow \pi_1(M)$  is isomorphic. Further the orientability character (indicatrix) of  $M$  and  $M - L$  are the same. Finally the same holds for the ends-number.*

**Proof.** If  $M$  is orientable, then so is  $M - L$  by (2.10). Conversely if  $M$  is not orientable, there is a one-sided Jordan curve  $J$  in  $M$  (2.11). We can find a tube neighbourhood  $T$  for a sub-arc  $A \approx [0, 1]$  of  $L$  such that  $A \supset J \cap L$ .  $T$  is homeomorphic to a rectangle and we have a homeomorphism of triads  $(T, L \cap T, A) \approx ([-1, 2] \times [-1, 1], [0, 2] \times \{0\}, [0, 1] \times \{0\})$ . Using a self-homeomorphism of the rectangle which is the identity on the boundary and pushing the arc outside itself (and outside  $L \cap T$ ), its extension to  $M$  (by the identity outside  $T$ ) yields a “finger move” pushing  $J$  (plus its Möbius tubular neighborhood  $N$ ) outside  $L$  (enlarge  $A$  if necessary so that  $A \supset N \cap L$ ). So the configuration  $(N, J)$  is pushed into  $M - L$  showing its non-orientability.

The assertion regarding the  $\pi_1$  is proved by the same “finger move” trick. Indeed given a loop in  $M$  we may push it into  $M - L$  (noticing that the finger move homeomorphism is isotopic to the identity). Thus the natural morphism  $\pi_1(M - L) \rightarrow \pi_1(M)$  is onto. To get its injectivity, assume that  $[c]$  is in the kernel. So  $c$  is loop in  $M - L$  which is null-homotopic in  $M$ . Since the range of the homotopy is a compactum  $K$  we can find a subarc  $A$  of  $L$  large enough as to contain  $K \cap L$ , and a finger move push this outside  $L$  and produce a null-homotopy for  $c$  ranging through  $M - L$ . ■

## 2.10 Finitely-connected surfaces, cylinder ends (Kerékjártó)

With loose conventions, we could define the *connectivity* of a surface as the rank of its  $H_1$  with integer coefficients. This conflicts slightly with the classical Riemann-Betti convention, where simple-connectivity really corresponds to rank zero (not one!). So eventually just employ:

**Definition 2.26** The *rank* of a surface (metric or not) is its first Betti number, i.e. the rank of the first (singular) homology group  $H_1(M, \mathbb{Z})$ . When finite, say the surface to be of *finite-connectivity*.

The following trick of Kerékjártó 1923 [26] (only a baby case of his more general classification of all open metric surfaces) is quite foundational (equivalent to the classification of 1-connected metric surfaces) and pivotal subsequently:



**Theorem 2.27** (*Kerékjártó 1923*) *A metric surface of finite-connectivity is homeomorphic to a finitely-punctured closed surface. The latter closed model is uniquely defined, and consequently finitely-connected surfaces are classified by the connectivity (=rank of  $\pi_1$ ), the number of ends  $\varepsilon$  and the indicatrix. In particular, open metric surfaces of finite-connectivity possess an end neighbourhood homeomorphic to a punctured plane.*

**Proof.** (Via the classification of 1-connected surfaces (2.5), and some dirty tricks.—Hausdorff would say: *Ich mache Komplexe mit Komplexen!*) If the surface  $M$  is compact there is nothing to prove (2.4). Otherwise,  $\pi_1(M)$  is free (2.7), and  $H_1$  has finite rank. Fix a finite generating system  $a_1, \dots, a_r$  of  $\pi_1(M)$ . By regular neighborhood theory, any compactum in a PL-manifold is contained in a finite bordered sub-manifold. (This goes back to Whitehead, cf. e.g. Rourke-Sanderson as quoted in Nyikos [35].) This applies to metric surfaces by Radó's triangulations (2.3). Representing the  $a_i$  by loops  $c_i$ , we may cover the ranges of the  $c_i$  by a compact bordered subsurface  $W \subset M$ . By construction  $\varphi: \pi_1(W) \rightarrow \pi_1(M)$  is epimorphic. Using the kernel killing procedure (2.20), one can arrange  $\varphi$  to be isomorphic (controlling compactness as we only need to kill finitely many primitive elements of the kernel). (Following Nyikos [35], we could say that  $W$  is a bag for  $M$ .)

**Claim 2.28** *Let  $W$  have  $n$  contours (=boundary components), then we claim (and prove clumsily below) that  $M - W$  has also  $n$  components  $\varepsilon_i$ , whose closures  $\overline{\varepsilon_i}$  are non-compact bordered surfaces with one contour.*

**Proof of Claim.** Indeed since  $W$  is a bordered surface, each contour of  $\partial W$  has a collar and therefore is two-sided in  $M$ . Choose the collar-sides lying outside  $W$ . If two contours get connected outside  $W$  in  $M$ , then we can construct (by aggregating an outer connection with an inner connection inside  $W$ ) a loop in  $M$  whose intersection number (in homology mod 2) with both contours is 1, and therefore which cannot be homotoped into  $W$  (violating the surjectivity of  $\varphi$ ). Thus  $M - W$  has at least  $n$  components (and of course cannot have more). Further each residual piece  $\overline{\varepsilon_i}$  cannot be compact, otherwise by classification (2.4) jointly with Seifert-van Kampen some alteration at the  $\pi_1$ -level would be detected. For instance if  $\overline{\varepsilon_i}$  is a disc, some loop in  $W$  trivializes in  $M$ , violating the injectivity of  $\varphi$ , and if  $\overline{\varepsilon_i}$  is a complicated pretzel with one contour, then again by looking at an appropriate intersection number (with a fixed curve) gives a loop in  $M$  which cannot be homotoped to  $W$ .

**Back to the proof of 2.27.** When capping-off  $\overline{\varepsilon_i}$  by a disc we obtain the open surface  $\overline{\varepsilon_{icap}}$ , which punctured is homeomorphic to  $\varepsilon_i$ . Thus by (2.23), the rank  $\text{rk}H_1(\varepsilon_i) = \text{rk}H_1(\overline{\varepsilon_{icap}}) + 1$ . Aggregating just one piece, say  $\overline{\varepsilon}$ , of the decomposition  $M = W \cup (\bigcup_{i=1}^n \overline{\varepsilon_{icap}})$ , the Mayer-Vietoris sequence

$$\underbrace{H_2(W \cup \overline{\varepsilon})}_0 \rightarrow \underbrace{H_1(W \cap \overline{\varepsilon})}_Z \rightarrow H_1(W) \oplus H_1(\overline{\varepsilon}) \rightarrow H_1(W \cup \overline{\varepsilon}) \rightarrow H_0(W \cap \overline{\varepsilon}) \rightarrow H_0(W) \oplus H_0(\overline{\varepsilon}),$$

whose the last arrow is injective, truncates as  $H_1(W \cup \overline{\varepsilon}) \rightarrow 0$ . So  $\text{rk}H_1(W \cup \overline{\varepsilon}) = \text{rk}H_1(W) + \text{rk}H_1(\overline{\varepsilon}) - 1$ . Hence by induction,  $\text{rk}H_1(W \cup \bigcup_i \overline{\varepsilon_i}) = \text{rk}H_1(W) + \sum_{i=1}^n \text{rk}H_1(\overline{\varepsilon_i}) - n$ . By construction  $\text{rk}H_1(W) = \text{rk}H_1(M)$ , so  $\sum_{i=1}^n \text{rk}H_1(\overline{\varepsilon_i}) = n$ . Since a bordered surface with  $H_1 = 0$  having a unique compact contour is compact (cf. [13, Lemma 10]) it follows from (2.28) that  $\text{rk}H_1(\overline{\varepsilon_i}) \neq 0$ . Hence  $\text{rk}H_1(\overline{\varepsilon_i}) = 1$  for all  $i$ , and  $\text{rk}H_1(\overline{\varepsilon_{icap}}) = 0$ . Thus by freeness (2.7),  $\pi_1(\overline{\varepsilon_{icap}}) = 0$ , and by the classification (2.5) it follows that  $\overline{\varepsilon_{icap}} \approx \mathbb{R}^2$ . It remains now only to compactify each end  $\varepsilon_i$  by adding the point at infinity giving us the searched closed surface. This shows the first clause.

The third (last) clause is a trivial consequence of the first.

Finally, the second clause follows from the classification of closed surfaces (2.4). Indeed if  $M$  has  $n$  ends it is—by the first clause—homeomorphic to  $F_{n*}$ , a closed model  $F$  affected by  $n$  punctures. Comparing the characteristic of  $M$  with that of “its” closed model  $F$ , we have the following; e.g., remove from  $F$  small discs about the  $n$  punctures to get a bordered surface  $W$ , to which  $M$

retracts by deformation (hence  $\chi(M) = \chi(W)$ ), and which has lost  $n$  2-simplices w.r.t.  $F$  (hence  $\chi(W) = \chi(F) - n$ ):

$$\chi(M) = 1 - b_1 + b_2 = \chi(F) - n. \quad (3)$$

Now assuming (an unfortunate) general collapse of our optical systems, with our brain-memories only able to remind from  $M$  its numerical invariants of connectivity  $b_1$  (rank), indicatrix and ends-number  $n$ , the above formula (where  $b_2 = 0$  as soon as  $M$  is open) determines  $\chi(F)$  (of “the” compact model) uniquely. Since puncturing finitely many points does not affect the indicatrix of a surface (2.11)—though it may well do so for a non-Hausdorff curve (e.g., *lasso!*)—the indicatrix of  $F$  is prescribed by that of  $M$ . Thus by the compact classification (2.4) the topology of  $F$  is unambiguously determined, and so is the topological type of  $M$ . (Recall that the group of self-homeomorphisms of a manifold acts transitively over finite configurations for any prescribed cardinality.) ■

Here are two examples that will play a special rôle in the foliated sequel:

**Lemma 2.29** *A dichotomic metric surface with  $\pi_1 = F_2$  is  $S_{3*}^2$  (thrice punctured sphere).*

**Proof.** Having finite-connectivity, the surface  $M$  is, by Kerékjártó (2.27), a punctured closed surface  $F_{n*}$ . Since dichotomic implies orientable (2.14), the closed model  $F$  is orientable (2.11) hence  $F \approx \Sigma_g$  (sphere with  $g$  handles). Since  $\pi_1 = F_2$  is not the group of a closed surface,  $M$  is open, and so  $b_2 = 0$ . Thus  $\chi = 1 - b_1 = 2 - 2g - n$ . Since  $b_1 = 2$ , we have  $2g + n = 3$  implying (as  $g, n \geq 0$ ) that  $(g, n) = (0, 3)$  or  $(1, 1)$ ; the latter option being precluded by dichotomy. ■

**Lemma 2.30** *A non-orientable metric surface with  $\pi_1 = F_2$  is  $\mathbb{R}P_{**}^2$  or  $\mathbb{K}_*$  (twice-punctured projective plane or once-punctured Klein bottle).*

**Proof.** Being of finite-connectivity, the surface  $M$  is, by Kerékjártó (2.27), a punctured closed surface  $F_{n*}$ . Since orientability is hereditary to subregions (2.10), the closed model  $F$  is non-orientable, hence  $F \approx S_{gc}$  (sphere with  $g \geq 1$  cross-caps). Since  $\pi_1 = F_2$  is not the group of a closed surface,  $M$  is open, and so  $b_2 = 0$ . Thus  $\chi = 1 - b_1 = 2 - g - n$ . Since  $b_1 = 2$ , we have  $g + n = 3$  implying (as  $g \geq 1, n \geq 1$ ) that  $(g, n) = (1, 2)$  or  $(2, 1)$ . ■

## 2.11 The soul of a non-metric finitely-connected surface (Kerékjártó, Nyikos)

One can imagine that any surface of finite-connectivity has a metric soul capturing its salient topological features and outside which nothing more happens. This reminds the phraseology “*the garbage must cease*” coined in Nyikos 1984 [35]. In the  $\omega$ -bounded case (which implies finite-connectivity cf. e.g., [15]) the above desideratum is a weak form of the bagpipe theorem of Nyikos 1984 [35]. Thus the present soul is merely a non-metric version of Kerékjártó cylindrical ends theorem (2.27) as well as an extension of Nyikos’ bagpipe (at any rate a typically Hungarian endeavor.)

As we are doing 2D-topology, the God-given recipe to capture metrically the whole topology is to impose “incompressibility” at the fundamental group level:

**Definition 2.31** *A soul  $S$  for a (non-metric) surface  $M$  of finite-connectivity is a metric subregion  $S \subset M$  such that the morphism induced by inclusion  $\varphi: \pi_1(S) \rightarrow \pi_1(M)$  is isomorphic.*

*Existence* of a soul is immediate from the kernel killing procedure (2.20), and the interesting issue is *uniqueness* (up to homeomorphism):

**Theorem 2.32** *Let  $M$  be a (non-metric) surface of finite-connectivity. Then the three characteristic invariants  $(\chi, \varepsilon, a)$  (viz. Euler characteristic, number of ends and indicatrix  $a = 0, 1$  whether orientable or not) of a soul are uniquely defined by the whole surface  $M$ , coinciding with those of  $M$ . Consequently:*

- (a) *The topological type of a soul is uniquely defined and referred to as the soul of the finitely-connected surface  $M$  (apply Kerékjártó (2.27)).*
- (b) *Any finitely-connected surface has a finite number of ends (an issue not completely obvious a priori).*

**Proof.** If  $M$  is compact this adds nothing new to the classical classification (2.4). So assume  $M$  open and then  $b_2 = 0$  (by vanishing of the top-dimensional homology, cf. e.g., Samelson [39]), so that  $\chi = 1 - b_1$ . Hence the knowledge of  $\chi$  is equivalent to that of the connectivity  $b_1$ . Hence the matching of  $\chi$  is immediate from the *soul*-definition (2.31).

The equality of the indicatrix (telling us orientability) is evident as well. For instance one can use the canonical group-morphism  $\mu_M: \pi_1(M) \rightarrow \{\pm 1\}$  obtained by propagating local orientation around loops (2.9). Orientability ( $a = 0$ ) amounts to the triviality of  $\mu_M$ . Since we naturally have  $\mu_M \varphi = \mu_S$ , equality of the indicatrix follows since  $\varphi$  of (2.31) is isomorphic.

It remains only to check the equality of the ends-number. We recall its:

**Definition 2.33** The *ends-number* of a space  $X$  is the maximal cardinality of non-relatively-compact residual components of a compactum  $K$  of  $X$ :

$$\varepsilon(X) = \sup_{K \text{ cpc} \subset X} \text{card}\{C \in \pi_0(X - K) : \overline{C} \text{ is non-compact}\}$$

—*Example:* Consider a letter “Y” with 3 branches going to infinity. Choose as compactum a point right below the branching, we count 2 residual components, but enlarging it we get 3 residual components (and never more!). The space has 3 ends.

**Step 1 (Deriving from a soul a weak bag-pipe decomposition).**

Given a soul  $S$  of  $M$  (hence of finite-connectivity), we know that it is homeomorphic to  $F_{n*}$  a finitely  $n$  times punctured closed surface  $F$  (2.27). Thus there is a compact bordered subsurface  $B \subset S$  obtained by removing from  $F$  the interior of little discs centered at the punctures. We call  $B$  a bag. It is a retract-by-deformation of the soul  $S$ , thus having the same  $\chi$ , a number of contours equal to  $n$  and the same indicatrix.

If we remove the interior of the bag  $B$  from  $S$  we have  $n$  residual components. Thus removing  $\text{int} B$  from  $M$  gives  $k \leq n$  components  $P_1, \dots, P_k$  which are bordered surfaces with  $d_i \geq 1$  contours. Note that  $\sum_i d_i = n$ .

First we claim that  $k = n$  and that all  $P_i$  are non-compact, for otherwise arguing as in Claim 2.28 violates the isomorphism of  $\varphi': \pi_1(B) \rightarrow \pi_1(M)$  (incompressibility condition). It follows that  $d_i = 1$  for all  $i$  (all  $P_i$  have a single contour).

Next using the Mayer-Vietoris sequence we have the following additivity relation (intuitively the overlapping occurs along circles not contributing to  $\chi$ ):

$$\chi(M) = \chi(B) + \sum_{i=1}^n \chi(P_i). \quad (4)$$

Since each  $P_i$  is bordered (and connected),  $b_2(P_i) = 0$ , and so  $\chi(P_i) = 1 - b_1(P_i) \leq 1$ . If  $b_1(P_i) = 0$ , then as  $P_i$  has a single contour it follows that  $P_i$  is compact (cf. [13], Lemma 10), an absurdity. Hence  $\chi(P_i) \leq 0$ , and since  $\chi(M) = \chi(B)$  it follows from (4) that  $\chi(P_i) = 0$  for all  $i$ . Thus the filled  $P_i$ , denoted  $P_{i, \text{filled}}$  (defined by gluing a disc to its unique contour), is a 1-connected surface (as its  $\chi = \chi(P_i) + 1 = 1$ , so its  $b_1 = 0$ , and as  $\pi_1$  is free (3.1) its  $\pi_1 = 0$ ). (In Nyikos’ jargon the  $P_i$  now truly deserve the name of *pipes*, yet not necessarily *long pipes* which terminology might be reserved to the  $\omega$ -bounded case).

**Step 2 (Computing the ends-number)** Since  $M$  contains the bag  $B$  (as a compactum) leaving  $n$  residual (non-relatively-compact) components (cf. Step 1), we have  $\varepsilon(M) \geq n$ . Conversely given a compactum  $K \subset M$ , we may decompose it according to the bagpipe decomposition  $M = B \cup \bigcup_{i=1}^n P_i$  to obtain

a fragmentation  $K = K_B \cup \bigcup_{i=1}^n K_i$ , where  $K_B = K \cap B$  and  $K_i = K \cap P_i$  which are all compacta (recall the bag and the pipes to be bordered hence closed as point-sets). Regarding each  $K_i$  in the filled pipe  $P_{i, \text{filled}}$ , we can trace a Jordan curve  $J_i$  containing  $K_i$  in its interior, cf. (2.34) below. Since the interior  $U_i$  of  $J_i$  is homeomorphic to an open-cell (or the plane) which is one-ended  $U_i - K_i$  has exactly one component which is not relatively-compact. Reconstructing the manifold  $M$  from its bagpipe structure it follows that  $M - K$  has *at most*  $n$  components, which are not relatively-compact. (Some “percolation” of the connectedness may of course occur within the bag.) This shows that  $\varepsilon(M) \leq n$ , completing the proof. ■

**Lemma 2.34** *Any compactum  $K$  of an open simply-connected surface  $M$  can be “enclosed” in a Jordan curve  $J$ , in the sense that the bounding disc for  $J$  (given by Schoenflies (2.6)) contains  $K$  in its interior.*

**Proof.** By calibration (2.21), we have an exhaustion of  $M$  by Lindelöf subregions  $M_\alpha$  with trivial groups  $\pi_1(M_\alpha) \approx \pi_1(M) = 0$ . Thus  $M_\alpha$  is  $S^2$  or  $\mathbb{R}^2$  (2.5). In the sphere case  $M_\alpha$  is both closed and open, hence equal to  $M$  (connectedness), violating the openness assumption. Thus  $M_\alpha$  is the plane for all  $\alpha$ . Since  $K$  is compact, there is  $\beta < \omega_1$  such that  $M_\beta \supset K$ . By Heine-Borel,  $K$  is closed and bounded, hence contained in a ball of large radius. ■

### 3 Algebraic distractions (Freiheitssätze)

#### 3.1 Freeness of the fundamental group of open surfaces (Ahlfors-Sario)

It is well known that the fundamental group of an open metric surface is free on countably many generators (2.7). Using Whitehead’s spine (2.8) we get the stronger assertion that such surfaces are homotopy equivalent to a countable graph. In general, a *non-metric* (Hausdorff) surface may well deliver a free fundamental group requiring uncountably many generators, as for the doubled Prüfer surface  $2P$  (cf. Calabi-Rosenlicht [8, p.339–40] for a complicated(?) proof of the non-denumerability of  $\pi_1(2P)$  or Gabard 2008 [12, Prop. 3] for an easy computation via Seifert-van Kampen, which was suggested by M. Baillif). It puzzled us, over a long period of time, whether the fundamental group of an arbitrary (non-metric) open surface is free (e.g., both [12, p.272] and Baillif 2011 [2] raise this question), yet it is probably a trivial exercise. The basic idea is that if there is a relation in the  $\pi_1$  of the (big) non-metric surface then, covering by charts the range of a null-homotopy materializing this relation, we get a Lindelöf subregion where this relation holds already, violating the freeness in the metric case. The trick looks theological, as it does *not* exhibit a basis for the fundamental group. Let us look if this naive idea can be completed to a serious argument.

**Proposition 3.1** *The fundamental group of any open surface is a free group.*

**Proof.** Let  $M$  be any open surface. If  $G := \pi_1(M)$  is not free then there is a reduced non-empty word  $w = w(x_1, \dots, x_k)$  in some variables  $x_i$  which purely specialized to elements  $g_i \in G$  yields the equation  $w(g_1, \dots, g_k) = 1$  in  $G$  (cf. Lemma 3.2 below and the definition after it for the meaning of pureness). Choose  $c_i$  some loops representing the  $g_i$ . Cover the range of a null-homotopy shrinking the concatenation  $w(c_1, \dots, c_k)$  to the basepoint by a finite number of charts to obtain a Lindelöf subregion  $L$ . Of course  $L$  contains the  $c_i$  (their ranges to be accurate), and so the  $c_i$  define elements in  $\pi_1(L)$ , say  $\gamma_i$ . Of course the relation  $w(\gamma_1, \dots, \gamma_k) = 1 \in \pi_1(L)$  continues to hold and the  $\gamma_i$  are all non-trivial since they map to the  $g_i \neq 1$  under the morphism  $\pi_1(L) \rightarrow \pi_1(M)$  induced-by-inclusion. Notice that the specialisation of  $w$  via the assignment

$x_i \rightarrow \gamma_i$  is pure. By the reverse implication of Lemma 3.2 we deduce that  $\pi_1(L)$  is not free, violating the classical (metric) case of the proposition (2.7). ■

The next lemma sounds tautological: *a group is not free iff there is a relation*, and just amounts to the interplay between the universal description of free groups with the more concrete model in terms of words spelled in an alphabet:

**Lemma 3.2** *A group  $G$  is not free if and only if there is a non-empty reduced word  $w = w(x_1, \dots, x_k)$  in  $k \geq 1$  letters  $x_1, \dots, x_k$  and a pure specialization  $x_i \rightarrow g_i$  to elements  $g_i \in G$  (cf. definition below) such that  $w(g_1, \dots, g_k) = 1$ .*

**Definition 3.3** A specialisation of a word  $w$  in a group  $G$  is *pure* if whenever two letters  $x, y$  of  $w$  are adjacent they do not specialize on  $g \in G$  and  $g^{-1}$ , and if  $xy^{-1}$  or  $x^{-1}y$  appears in the word  $w$ ,  $x$  and  $y$  do not specialize on the same element  $g \in G$ . We also demand that no letter of  $w$  specialize to  $1 \in G$ .

**Proof.** [ $\Rightarrow$ ] If  $G$  is not free, then  $G$  is still the quotient of a free group  $\varphi: F \rightarrow G$  with non-trivial kernel  $\ker \varphi$ . Pick a non-trivial element  $1 \neq x \in \ker \varphi$ . Let the set  $X$  be a basis for the free group  $F$ . As is well-known  $X$  generates  $F$  and  $x \in F$  can be written as a non-empty reduced word  $w = w(x_1, \dots, x_k)$  involving finitely many  $x_i \in X$ . Let  $g_i = \varphi(x_i)$ . As  $\varphi(x) = 1$ , we have  $w(g_1, \dots, g_k) = 1$ . Furthermore pureness of the specialisation  $x_i \rightarrow g_i$  follows, if we take care assuming the word  $x$  to have minimal length among all those non-trivial elements of the kernel  $\ker \varphi$ .

[ $\Leftarrow$ ] Assume that  $G$  is free, say with basis  $X \subset G$ . Let  $w = w(x_1, \dots, x_k)$  be a non-empty reduced word in some abstract symbols  $x_i$  and let  $x_i \rightarrow g_i \in G$  be a pure specialization; to show  $w(g_1, \dots, g_k) \neq 1$ . Each  $g_i$  can be written as a non-empty reduced word  $w_i$  involving finitely many letters of the alphabet  $X$ . Substitute these expressions in  $w$  to obtain the big word  $w(w_1, \dots, w_k)$ . If the latter collapses completely under reduction then this forces two adjacent words  $w_i, w_j$  to cancel out, violating the pureness of the specialization. ■

### 3.2 Freedom for non-Hausdorff 1-manifolds by reduction to surfaces

Another “mystical” question (also eluding us for a long time, and indeed still eluding us slightly) is whether the fundamental group of a (non-Hausdorff) 1-manifold is always free. (For Hausdorff 1-manifolds, we have a classification in 4 specimens, which all have trivial groups, except the circle.) Of course the same Lindelöf reduction as we just did for Hausdorff surfaces is formally possible, yet not very effective unless the Lindelöf case is settled.

Maybe the royal road (suggested by a discussion with A. Haefliger) to this freeness curiosity is a geometric construction exhibiting any (non-Hausdorff) 1-manifold  $M^1$  as the base of a fibration of a Hausdorff surface  $M^2$  by real-lines (recovering via the leaf-space the given  $M^1$ ). The exact homotopy sequence of a fibration<sup>8</sup> reads (denoting by  $F \approx \mathbb{R}$  the fibre):

$$\{1\} = \pi_1(F) \rightarrow \pi_1(M^2) \rightarrow \pi_1(M^1) \rightarrow \pi_0(F), \quad (5)$$

from which we deduce the required freeness of  $\pi_1(M^1)$  via (3.1). In the simplest case where  $M^1$  is a branching line or a line with two origins it is clear how to construct such a “thickened” fibration  $M^2 \rightarrow M^1$ , essentially like for *train-tracks* (à la Thurston-Penner), cf. Figure 2. The above idea originates in Haefliger, 1955 [17, p. 8, point 2.], where we read:

On peut montrer que toute variété à une dimension avec un nombre fini de bord et dont le groupe fondamental a un nombre fini de générateurs<sup>9</sup>, est l'espace des feuilles d'une structure feuilletée, et même la base d'une fibration par des droites définie sur une variété séparée à deux dimensions.

<sup>8</sup>This foolhardy idea was suggested orally by Haefliger (circa 2006). One has to convince that the classical proof does not use the Hausdorffness of the base; compare Hopf-Eckmann, Ehresmann-Feldbau, Steenrod, etc., yet a detailed redaction is maybe desirable.

<sup>9</sup>Of course we may wonder if this proviso is really required. We believe it is not.

It also reappears in Haefliger-Reeb 1957 [18, p.125, last sentence], where it is asserted (again without proof) that any second-countable 1-connected (non-Hausdorff) 1-manifold can be realised as the leaf-space of a suitable foliation of the plane.

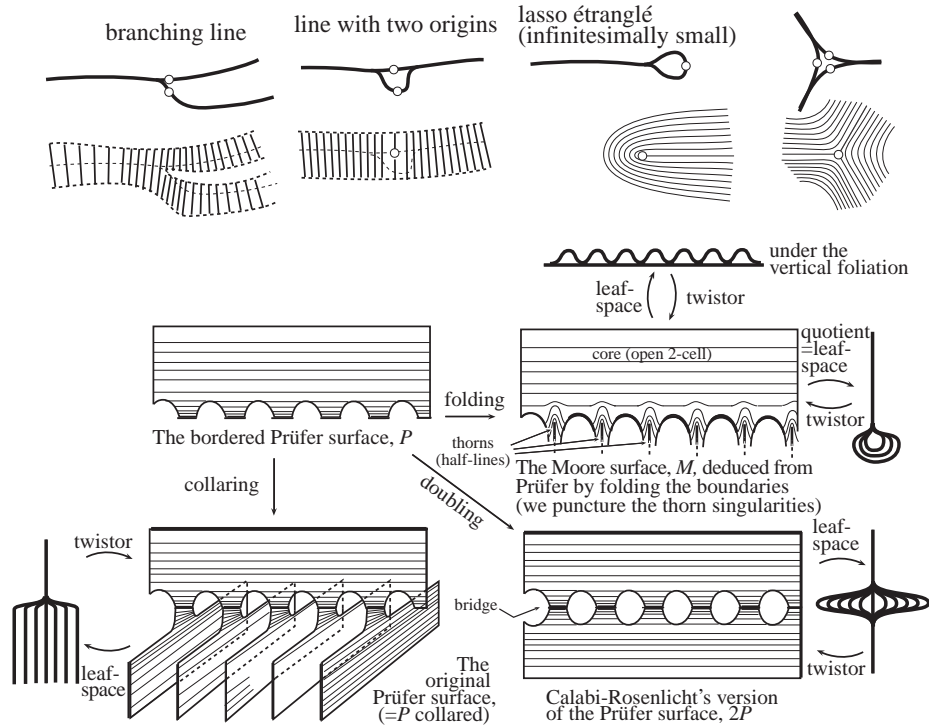


Figure 2: The Haefliger twistor of a (non-Hausdorff) 1-manifold

### 3.3 Twistor or train-tracks (Haefliger, Thurston, Penner)

Can somebody prove the following *hypothetical* lemma (strongly inspired from Haefliger and Thurston, Penner's train-tracks):

**Lemma 3.4** (*Twistor trick or train-tracks.*) *Any (non-Hausdorff) 1-manifold can (non-canonically) be materialized as the base of a fibration  $p: M^2 \rightarrow M^1$  of a Hausdorff surface fibred by real-lines  $\mathbb{R}$ . In particular, the projection  $p$  induces an isomorphism on the  $\pi_1$ .*

We hope the result is true, being implicitly used in Haefliger-Reeb [18] at least when the manifold  $M^1$  is 1-connected and second-countable (equivalently Lindelöf, because manifolds are locally second-countable). Another little piece of evidence is that a (Morse theoretical) reverse engineering seems to hold metrically: *any open metric surface can be fibred by lines so that the quotient is a non-Hausdorff curve* (4.10). Even if (3.4) should work only in the Lindelöf case, this would be punchy enough to settle the general freeness question (in view of the Lindelöf reduction trick used in (3.1)).

**Example 3.5 (Non-Lindelöf twistors)** It is worth noticing that Prüfer's construction (and its derived products like Moore or Calabi-Rosenlicht) provides twistors for several toy-examples of non-Lindelöf 1-manifolds (cf. Figure 2, bottom part). For instance the horizontally foliated (classical) Prüfer surface twistorize the line with continuously ( $\mathfrak{c} = \text{card}(\mathbb{R})$ ) many branches. (Hence there is at least no visceral incompatibility between the twistor desideratum (3.4) and the non-Lindelöf context.) Likewise the leaf-space of the horizontally-foliated doubled Prüfer surface  $2P$  is the line with  $\mathfrak{c}$ -many origins. The vertical foliation on the Moore surface punctured along the folded points admits as leaf-space

(and therefore is a twistor for) the *everywhere doubled line* (described in Baillif-Gabard 2008 [3, §3]). Finally, the horizontally-foliated Moore surface punctured at all thorns singularities is a twistor for the “*lasso étranglé*” with continuously many (infinitesimal) loops.

We now tabulate formal consequences of this geometric construction (3.4):

**Corollary 3.6** (*Haefliger-Reeb 1957 [18]*) *All simply-connected second-countable (non-Hausdorff) 1-manifold occur as the leaf-space of a suitable foliation of the plane. Relaxing second-countability of  $M^1$ , the same is true for a suitably foliated simply-connected surface.*

**Proof.** Given such a 1-manifold  $M^1$ , we consider its twistor  $\tau M^1 \rightarrow M^1$  given by (3.4). By the isomorphism (5), the total space  $M^2 := \tau M^1$  is simply-connected and Lindelöf (by the general topology version of Poincaré-Volterra, cf. Bourbaki or Guenot-Narasimhan as referenced in [14]). Since  $M^2$  is non-compact (containing lines as closed subsets), it is homeomorphic to  $\mathbb{R}^2$  by classification of 1-connected surfaces (2.5). ■

More generally we have the following (with relaxable parenthetical provisos):

**Corollary 3.7** *Any (second-countable) 1-manifold is the leaf-space of a foliation by lines of a (metric) surface with the same fundamental group.*

Finally regarding the fundamental group structure we have:

**Corollary 3.8** *All (non-Hausdorff) 1-manifolds have free fundamental groups.*

**Proof.** Consider again the twistor  $M^2 \rightarrow M^1$  of the 1-manifold. By (5) again,  $\pi_1(M^2)$  is isomorphic to  $\pi_1(M^1)$ , and the former is free by (3.1). In case the twistor trick (3.4) should hold only for Lindelöf 1-manifolds, then first establish the corollary in that case, and next extend universally by the Lindelöf reduction trick used in the proof of (3.1). ■

## 4 Foliated foundations

Before penetrating truly to our main object, we recall some classical facts for later references. Below, 1-*foliation* abbreviates “one-dimensional foliation”.

### 4.1 Orienting double cover (Haefliger, Hector-Hirsch, etc.)

**Proposition 4.1** *Given a 1-foliation of a manifold, there is a double cover such that the lifted foliation is orientable. In particular, any 1-foliation of a simply-connected manifold is orientable.*

**Proof.** If no smoothness is postulated, some tricks with germs act as a substitute to the tangent line bundle of the foliation (cf. Haefliger 1962 [19] or Hector-Hirsch 1981-83 [21, 22]). Since the construction is purely local, there is no hindrance in implementing it in the globalized world of non-metric manifolds. ■

### 4.2 Compatible flows (Kerékjártó, Whitney)

In contrast the following paradigm is much more *metric*-sensitive (indeed false without one, as amply discussed in [14]):

**Theorem 4.2** (*Kerékjártó 1925, Whitney 1933*) *Given an oriented 1-foliation of a metric manifold, there is a compatible flow, whose trajectories are the leaves.*

**Proof.** The 2D-case is due to Kerékjártó 1925 [27], and the general one to Whitney 1933 [43]. ■

**Corollary 4.3** *The 2-disc cannot be foliated (tangentially).*

**Proof.** Recall two classical arguments:

- *Via Brouwer.* Assuming it could, then as the disc is 1-connected the foliation is orientable (4.1), hence admits a compatible flow (4.2). Passing to dyadic times  $t_n = 1/2^n$  of the flow, gives a nested sequence of non-empty closed sets (Brouwer’s fixed point theorem) whose common intersection is non-void by compactness. Thus a rest-point for all times is created, violating the flow compatibility with the foliation.

- *Via H. Kneser.* Double the foliated disc to get a foliated sphere with  $\chi = 2$ , violating Kneser’s combinatorial proof of the Euler obstruction (4.7) below. ■

**Corollary 4.4** *More generally, a closed topological manifold foliated by curves has zero Euler characteristic.*

**Proof.** If not, then passing to the orienting cover (4.1), we may assume the foliation oriented. Consider a compatible flow (4.2), which by Lefschetz’s fixed point theorem [29] (version for ANR’s) has a fixed point, an absurdity. In the surface case one can also argue elementary à la Kneser via (4.7). ■

### 4.3 Beck’s technique (plasticity of flows)

Albeit we are primarily interested in foliations, some facts concerning flows will be useful in the sequel. A basic desideratum, when dealing with flows, is a two-fold yoga of “restriction” and “extension”:

- (1) *Given a flow on a space  $X$  and an open subset  $U \subset X$ , find a flow on  $U$  whose phase-portrait is the trace of the original one; and conversely:*
- (2) *Given a flow on  $U$ , find a flow on  $X \supset U$  whose phase-portrait restricts to the given one.*

Thus, one expects that any open set of a brushing<sup>10</sup> is a brushing, and that any separable super-space of a transitive space is transitive, provided the sub-space is dense (or becomes so, after a suitable inflation).

Problem (1) is solved in Beck [6], when  $X$  is metric (via passage to the induced foliation this also derives from Kerékjártó-Whitney (4.2)). (An example in [14] indicates a non-metric disruption.) The same technique of Beck (clever time-changes afforded by suitable integrations), solves Problem (2) in the metric case (compare [25, Lemma 2.3]):

**Lemma 4.5** *Let  $X$  be a locally compact metric space and  $U$  an open set of  $X$ . Given a flow  $f$  on  $U$ , there is a new flow  $f^*$  on  $X$  whose orbits in  $U$  are identic to the one under  $f$ .*

### 4.4 Foliated triangulations (H. Kneser)

It is hard to resist recalling Hellmuth Kneser’s combinatorial approach to the Euler obstruction. We admit the following referring for clean proofs to Kneser 1924 [28] or Hector-Hirsch [21].

**Lemma 4.6** (*Kneser 1924*) *Any metric foliated surface has a “generic” triangulation where each 2-simplex is transversely foliated as depicted on Fig. 3.b.*

**Proof.** (Dirty outline) By definition of a foliated structure it is rather clear that we have a tessellation by foliated boxes, which are squares. Then we add diagonals to get triangles and whenever two of them are adjacent along a piece of leaf, we perform Kneser’s flip depicted on Fig. 3.e (gaining transversality). ■

Since any open metric surface can be foliated (4.10) (=Morse theoretical trick), this suggests another proof of Radó’s triangulation theorem (2.3) at least for open surfaces. (Of course Radó was well aware of Kneser’s paper, cf. his article [37], but probably not of the Morse theoretical trick.)

<sup>10</sup>That is a space admitting a fixed-point free flow.



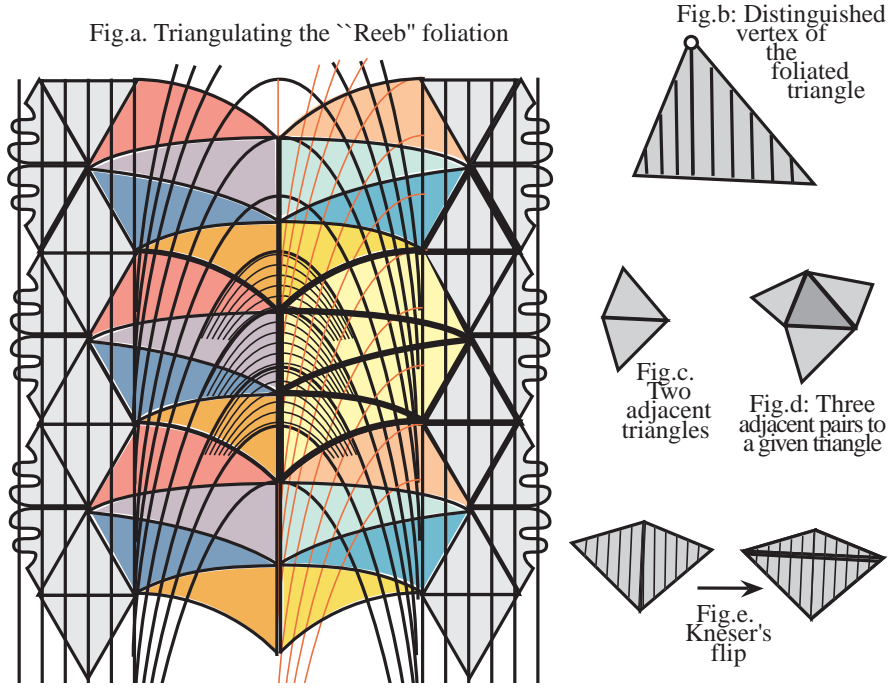


Figure 3: Kneser's proof of the Euler-Poincaré-Dyck obstruction

**Corollary 4.7** (*Poincaré 1885, Dyck 1888, Kneser 1924*) *A closed surface which is foliated has vanishing Euler characteristic  $\chi = 0$ .*

**Proof.** (Kneser). By (4.6) there is a triangulation transverse to the foliated structure, which is finite by compactness. So we may compute the characteristic as the alternating sum of the cardinalities  $e_i$  of the set  $\sigma_i$  of simplices of dimensionality  $i = 0, 1, 2$ :

$$\chi = e_0 - e_1 + e_2. \quad (6)$$

First ignoring the foliation, recall the relation  $2e_1 = 3e_2$ , cf. (4.8) right below. Besides, any (transversely foliated) 2-simplex has a distinguished vertex through which a piece of leaf traverses the 2-simplex (Fig. 3.b). So we have a map  $\sigma_2 \rightarrow \sigma_0$  which is onto and 2-to-1 as the leaf extends in 2 directions. Hence  $e_2 = 2e_0$ . Plugging those relations in (6) gives:  $\chi = e_0 - e_1 + e_2 = \frac{1}{2}e_2 - \frac{3}{2}e_2 + e_2 = 0$ . ■

**Lemma 4.8** (*Descartes, Euler 1750, L'Huilier 1811, who else?*) *In any finite triangulation of a closed (compact non-bordered) surface the relation  $2e_1 = 3e_2$  holds true between the numbers  $e_1, e_2$  of edges, respectively triangles.*

**Proof.** Let  $\sigma_i$  be the set of simplices of dimension  $i$  and consider the incidence relation *two triangles have a common edge* (adjacent triangles, Fig. 3.c):

$$I = \{(\Delta_1, \Delta_2) \in \sigma_2 \times \sigma_2 : \Delta_1 \cap \Delta_2 = \text{one edge}\}$$

Mapping such a pair to its common edge yields a map  $I \rightarrow \sigma_1$  which is onto (the surface being non-bordered) and 2-to-1 as the pair-order is permutable. Thus the cardinality of  $I$  is  $\#I = 2e_1$ . Besides, projecting on the first factor (say) gives a map  $I \rightarrow \sigma_2$  which is onto and 3-to-1 (Fig. 3.d), whence  $\#I = 3e_2$ . ■

#### 4.5 Open metric surfaces fibrates (Morse, Thom, etc.)

As well-known, metric differentiable manifolds admit *Morse functions*, which as a reaction to the complicated topology (or rather the compactness) deliver generally critical points. (In the late 60's, Morse as well as Kirby-Siebenmann explained how to get rid off the differentiable proviso using so-called *topological* Morse functions.) When the manifold is open, one can (in principle) eliminate critical points by rejection to  $\infty$ :

**Theorem 4.9** *Any open metric (topological) manifold has a critical point free Morse function. The latter, being a submersion, defines a codimension-one foliation, whose transverse “line-field” gives a 1-foliation whose leaves are lines.*

**Proof.** (*Heuristic outline*). Choose any Morse function  $f$ , i.e. locally resembling a quadratic non-degenerate form  $x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_n^2$ . In particular critical points are isolated. In every open metric manifolds  $M$ , one can—starting from any point—trace a *ventilator* (jargon borrowed from L. Siebenmann), i.e., an arc  $A$  homeomorphic to a semi-line  $[0, \infty)$  such that  $M$  slitted along  $A$  is homeomorphic to  $M$ , i.e.,  $M - A \approx M$ . Since  $M$  is metric, the critical-set of  $f$ , being discrete, is countable. Thus by a (hazardous?) infinite-repetition, we may remove inductively ventilators emanating from the critical points, to reach the first claim. The addendum follows by aping “the” gradient flow of the Morse function via a technique of Siebenmann [40]. (In the smooth case just integrate the transverse line-field, w.r.t. an auxiliary Riemannian metric.)

(*Variant of proof in the PL-case*) Compare Hirsch 1961 [23], using the theory of Whitehead’s spine. ■

**Proposition 4.10** *Any metric open surface foliates. More is true it can be foliated by lines, probably even in the following hygienical way:*

(*Hypothetical addendum*).—Any such surface can be regarded as the total space of a fibration by lines whose base is a (non-Hausdorff) 1-manifold.

**Proof.** (*High-brow proof*) This follows by specializing the above (4.9), taking (optionally) advantage of the smoothability of metric surfaces. Smoothing(s) can be deduced from Radó (2.3) using eventually the trick of Stoilow-Heins to introduce a (stronger) Riemann surface ( $\mathbb{C}$ -analytic) structure in the orientable case (and by adapting a Klein(=di-analytic) surface structure) in the non-orientable case.

(*Elementary proof?*) Start from a triangulation given by Radó (2.3) and try to find some clever combinatorial procedure to propagate a foliated texture à la Kneser. (Details left to the imaginative readers.)

—*Outlined addendum*. Integrating the vector field orthogonal to the level curves of a critical-point-free Morse function  $f$ , and having speed-one w.r.t. a complete Riemannian metric), we obtain a (fixed-point-free) flow  $\varphi: \mathbb{R} \times M \rightarrow M$  without “recurrences”. Let  $\mathcal{F}$  be the underlying foliation. The projection on the leaf-space  $p: M \rightarrow M/\mathcal{F}$  is a fibration (by lines). Indeed, given a trajectory of  $\varphi$  (say that of the point  $x$ ), one can let evolve in time a small 1D-chart  $V$  (selected) in the  $f$ -level-curve,  $f^{-1}(f(x))$ , through  $x$  to manufacture a 2-cell  $U := \varphi(\mathbb{R} \times V) \approx \mathbb{R} \times V$  (via  $\varphi(t, v) \leftarrow (t, v)$ ) which is (trivially) fibred by lines (the trajectories). The projection of  $U$  in the leaf-space is open (its inverse image being  $U$  which is open) and homeomorphic to  $V \approx \mathbb{R}$  (restrict  $p$  to  $V$ ). Hence the leaf-space is a 1-manifold, and  $p$  is a fibration (trivial over  $p(V)$ ). ■

This addendum looks somewhat dual to the engineering of Haefliger (3.4) permitting to conceive any (non-Hausdorff) 1-manifold as the base of a fibration by lines of a Hausdorff surface.

—*Baby example*. Consider a “Y”-shaped surface resembling a tree in usual 3-space  $\mathbb{R}^3$  with three trunks going to infinity (Fig. 4.a). The height function “ $z$ ” (third coordinate) has a critical point where the two branches of the tree “Y” bifurcate. The latter can be eliminated just by deforming one of the branch horizontally and letting it disappear to  $\infty$  like a “cusp” (Fig. 4.b). Those surfaces are just diffeomorphic to a punctured cylinder that can be imagined endowed with a complete Riemannian whose line-elements diminish in size (w.r.t. to the Euclidean element) as the puncture is approached (Fig. 4.c). The height-function is critical-point-free and the orthogonal trajectories are vertical lines on the cylinder-model (with a sole interruption at the puncture), yet drastically slowed-down (w.r.t. the Euclidean perception) as we approach the dark-matter concentrated near the puncture. The leaf-space (of the transverse foliation) is a circle-with two origins (just identify two transverse circles lying above resp. below the puncture, whenever they are intercepted by a same leaf) (Fig. 4.d).

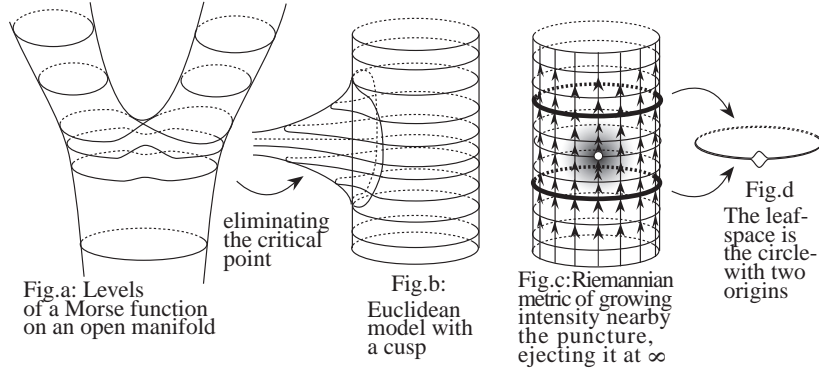


Figure 4: Foliating open surfaces by lines

## 5 Haefliger-Reeb theory for non-metric simply-connected surfaces

The starting point for this section (and actually of the whole paper) was the observation by D. Gauld that a foliated avatar of the “closing lemma” (Figure 5, Case 1) shows that a simply-connected surface (even when non-metric) cannot be transitively foliated. Exploiting this remark allows one to extend the Haefliger-Reeb theory to all 1-connected surfaces, pushing its validity outside any metrical predisposition, evidencing a rather robust character of their theory.

### 5.1 Haefliger-Reeb derives from Schoenflies (Gauld)

The game in this section is to ape non-metrically the Haefliger-Reeb theory 1957 [18] describing foliations on the plane  $\mathbb{R}^2$ . (This was also studied earlier in 1940 by W. Kaplan.) Replacing the plane by an arbitrary (non-metric) simply-connected surface, we passively observe that most of the classical theory remains valid in this broader context. (The only minor divergence is that the projection on the leaf-space can now cease to be a fibration, an issue only emphasised in subsequent papers, e.g. of Godbillon and Reeb.) The *raison d’être* for this extension is the non-metric availability of the Schoenflies theorem which is implied by, and indeed equivalent to, simple-connectivity (see Gabard-Gauld 2010 [13] or (2.6)).

**Proposition 5.1** *A foliated simply-connected surface satisfies:*

- (a) *Any leaf is open as a manifold (i.e., no compact circle leaf).*
- (b) *A leaf appears at most once in any foliated chart. More precisely if a leaf intersects a foliated chart then this intersection reduces to a single line (plaque).*
- (c) *From (b), it follows that the leaf-space is a 1-manifold, in particular any leaf is closed as a point-set. Also leaves are proper, i.e. the leaf topology matches with the relative topology. Still from (b) leaves cannot be dense.*
- (d) *By (a) any leaf has two ends and runs to infinity in both directions while dividing the surface in two components (called halves).*

**Remark 5.2** This statement may well be empty when the surface lacks any foliation. This is the case of the 2-sphere, but can also occur to non-compact surfaces, e.g. the long glass  $\mathbb{S}^1 \times \mathbb{L}_{\geq 0}$  capped off by a 2-disc (compare [4]). Point (b) is exactly Théorème 1 in Haefliger-Reeb [18, p.120], and a direct consequence is that the leaf-space is a (generally non-Hausdorff) 1-manifold.

**Proof.** (a) is obvious, for a circle leaf would bound a foliated disc by Schoenflies (see [13] or (2.6)), which is an absurdity (4.3).

The proof of (b) is a similar Schoenflies obstruction modulo some tricks reminiscent of the Poincaré-Bendixson trapping argument or rather the closing lemma (for dynamical flows). Assume that a leaf returns to a foliated chart. Orient the foliated box as well as the leaf. Then one distinguishes two cases

depending on whether the first-return to the box matches or reverses the orientation (cf. Figure 5). In fact since the foliation is orientable (4.1), only the first case needs attention.

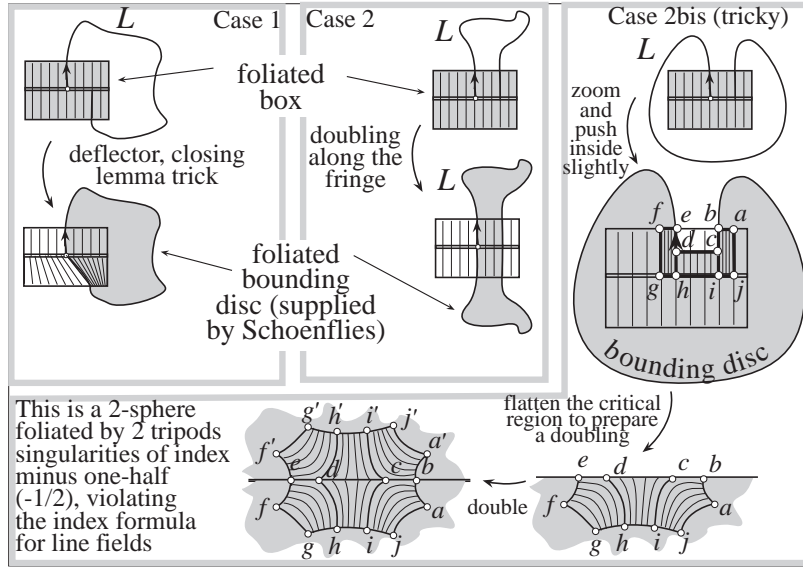


Figure 5: Absence of recurrences for a leaf in a simply-connected surface

In the first case one can perturb the foliation within the box (e.g., piecewise linearly) while creating a circle leaf (impossible by Schoenflies).

**Very optional over-exhaustive case distinctions.** In the other case one has a “tongue shape” whose double produces a foliated disc. In fact in the second case there is a tricky subcase (Case 2bis on Figure 5) corresponding to the situation, where the bounding disc for the Jordan curve starting from the first escapement  $e$ , say, of the oriented leaf  $L$  from the foliated-box  $B$  and extended until its first impact, say  $b$ , on the box  $B$  and closed-up by the unique arc in  $\partial B$  from  $b$  back to  $e$  (transverse to  $\mathcal{F}$ ) contains the foliated-box  $B$ . In this case we start by pushing the side  $\overline{eb}$  into the box up to position  $\overline{cd}$ . Then we flatten the boundary of the bounding disc  $D$  for the Jordan curve  $J$  (through  $e, b, c, d, e$ ) near the critical region (compare the figure), and finally we double  $D$  with a replica  $D'$  yielding a foliated 2-sphere having as unique singularities two “tripods” singularities located at the points  $c$  and  $d$ . Such tripods singularities have an index  $j = -\frac{1}{2}$  each, yielding a total sum of  $-1$ , disagreeing with the Euler characteristic of  $S^2$ . This violates the Poincaré-Kerékjártó-Hopf index formula for line fields (cf. e.g., H. Hopf [24, p. 109 and Theorem II, p. 113]).

(c) The properness of leaves is clear in view of (b). That leaves are closed sets can be derived from the fact that the leaf-space is a (non-Hausdorff) 1-manifold, which follows directly from (b), as we shall recall later (5.9). Of course closedness can be deduced also directly from (b): given a point  $p$  not on the leaf  $L$ , choose a foliated chart  $U$  about  $p$ . If  $U \cap L$  is empty we are done. If not then by (b) we see only a single plaque of  $L$  in  $U$  so that we easily find an open set  $V \subset U$  containing  $p$  but not intersecting  $L$ .

(d) As we shall not really need it, we leave as an exercise the task of clarifying the meaning of running to infinity (probably in terms of evasion from any compactum). The last claim of (d) is somewhat harder to establish (especially if one tries to delineate the broadest generality in which such a separation holds true). Thus we reserve the next section to a detailed discussion. ■

**Optional semi-historical digression.** The sequel may lead to an interpretation of the following prose of Haefliger-Reeb [18, p. 120]: “*Le théorème 1 qui suit est classique; sa démonstration repose sur le théorème de Jordan (dans une version particulièrement facile à établir); ...*” Of course Jordan is here somehow blended with Schoenflies. Recall incidentally that the nomenclature “Schoenflies theorem” for the bounding disc property is a rather recent coinage (perhaps first appearing in Wilder 1949, as noticed in Siebenmann 2005 [41, p. 651]). At any rate what is relevant to the sequel is that point (b) of Prop. 5.1 provides a local flatness allowing one to prove a version of Jordan separation using only covering space theory. Thus we match slightly with the version particularly easy to establish mentioned in Haefliger-Reeb, albeit they

probably rather had in mind a mod 2 homology argument, as shows the sequel of their text “...elle utilise donc essentiellement le fait que le plan  $\mathbb{R}^2$  est simplement connexe (ou plus précisément que son premier nombre de Betti modulo 2 est nul).” However their sketched proof of their Théorème 1 uses in fact Schoenflies and not merely Jordan separation (recall Dubois-Violette’s example).

## 5.2 Polarized covering à la Riemann and Jordan separation in the large

Given a hypersurface  $H$  in a simply-connected manifold  $M$ , it is intuitively clear that  $H$  divides  $M$ , provided the hypersurface is closed as a point-set. A possible strategy is that any such hypersurface in a manifold induces naturally a double cover of  $M$ . When  $H$  does not divide  $M$  this covering is connected, violating the simple-connectivity of  $M$ . This section details the above idea. First a:

**Definition 5.3** A (locally flat) hypersurface in a manifold is a (non-empty) subset  $H$  such that for any point  $p \in H$  there is an open neighbourhood  $U$  in  $M$  and a homeomorphism of triad  $h : (U, U \cap H, p) \approx (\mathbb{R}^n, \mathbb{R}^{n-1} \times \{0\}, 0)$ . Since  $U \cap H$  divides  $U$ , we call  $U$  a polarised chart. One has a splitting  $U = U_+ \cup U_-$  in two local halves defined as the closures in  $U$  of the components of  $U - (U \cap H)$ . In the sequel we shall refer to  $U_{\pm}$  as being semi-charts.

For instance the “open” straight line  $H = ]-1, +1[ \times \{0\}$  in  $\mathbb{R}^2$  is a hypersurface, but does not separate the plane. This is why we restrict attention to hypersurfaces, which are closed as point-sets. As the terminology “closed hypersurfaces” conflicts with the classical nomenclature “closed manifolds” (referring to compact borderless manifolds), some *ad hoc* jargon is coined to disambiguate the double usage of “closed” in point-set vs. combinatorial topology:

**Definition 5.4** A *divisor* in a manifold is a hypersurface in the sense of (5.3), whose underlying set is closed as a point-set.

Now “our” polarization trick is the following mechanism:

**Proposition 5.5** Given a divisor  $H$  in a manifold  $M$ , there is a naturally defined double cover  $M_H \rightarrow M$  (called the polarization of  $M$  along  $H$ ), with the distinctive property that  $M_H$  is disconnected if and only if  $H$  divides  $M$ .

**Proof. (1) Intuitive idea.** First we can imagine that we cut  $M$  along  $H$  to obtain a bordered manifold  $W$  with an involution  $\sigma$  on the boundary  $\partial W$  telling one how to reglue the points to remanufacture the manifold  $M$  out of  $W$ . (We use here the magic scissor of combinatorial topology, which instead of deleting points rather duplicate them!) In particular one has an *assembly* map  $\alpha : W \rightarrow M$ , which is one-to-one except over  $H$  where the fibers are two points exchanged by  $\sigma$ . (Call  $\sigma p$  the opposite of  $p$ .) Then take  $W'$  a replica of  $W$ , and denote by  $p' \in W'$  the twin copy of the point  $p \in W$ . In the disjoint union  $W \sqcup W'$  identify the point  $p \in \partial W$  with the opposite of its twin, i.e.  $\sigma p'$  (where for simplicity we still denote by  $\sigma$  the involution on  $\partial W'$ ). We define  $M_H$  as the resulting quotient space. It is not hard to show that the assembly maps  $\alpha \cup \alpha' : W \sqcup W' \rightarrow M$  induce a map  $M_H \rightarrow M$  which is a covering projection.

**(2) Another viewpoint.** The above construction requires a cleaner description of the cutting process. We can take a slightly different approach. First take a copy  $M'$  of  $M$ , and define a new topology by splicing any polarized chart  $U = U_+ \cup U_-$  (cf. Def. 5.3) into the two “spliced” sets  $U_+ \sqcup U'_-$  and  $U'_+ \sqcup U_-$ . The “primes” indicates that we push alternatively one of the two halves of  $U$  into the second layer  $M'$ . Further we would like to identify the points  $p$  in  $U \cap H = U_+ \cap U_-$  with their twins  $p'$  so as to restore the locally Euclidean character. Note that we are not merely redefining a new topology on the (static) point-set  $M \sqcup M'$ , but really doing a gluing on the two spliced charts which is easy locally, yet maybe problematic (at the non-metric scale).

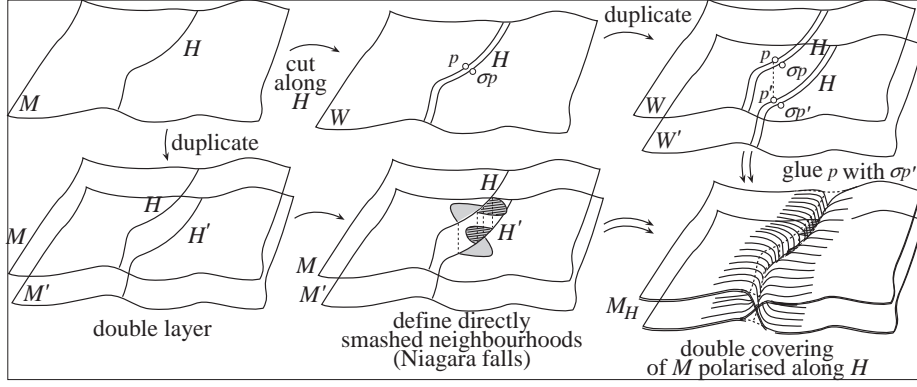


Figure 6: The double cover of a manifold  $M$  polarized along a hypersurface  $H$

**(3) Finding an issue.** Maybe the trick is as follows, closer to the approach (1). We would like to formalise the idea of cutting along a hypersurface. Thus we need first to enrich  $M$  by creating a replica for each point lying on  $H$ . We try to think of such a point as a pair  $(p, U_{\pm})$  consisting of a (classical) point  $p$  of  $H$  plus a preferred half  $U_{\pm}$  of a polarized chart  $U$  about  $p$ . Since  $H$  is closed (as a point-set), we may fix an atlas for  $M$  such that any chart meeting  $H$  is a polarised chart (first cover the hypersurface by polarized charts and then aggregate charts of the manifold  $M - H$ ). Say that such an atlas is polarised w.r.t.  $H$ . Given a polarised atlas  $\mathcal{A}$  (say a maximal one to kill any dependence upon anodyne choice from the beginning) we define a new point-set  $W$  as consisting of all *filters*, in the following two senses:

**Definition 5.6** A filter  $F$  of charts (resp. of semi-chart) is a nested sequences of charts  $U_i \supset U_{i+1}$  of  $\mathcal{A}$  (resp. semi-charts, i.e. halves of polarized charts of  $\mathcal{A}$ ) whose common intersection  $\bigcap_{0 \leq i \leq \omega} U_i$  is a unique point (called the center of the filter).

Declare two filters  $F_1, F_2$  as *equivalent* if for any member of the first  $U_i \in F_1$  there is an element of the second  $V_j \in F_2$  such that  $V_j \subset U_i$ . It is easy to check that this is an equivalence relation. Now define  $W$  as the set of equivalence classes of filters. Notice that there are two equivalence classes of filters converging to a point  $p \in H$ , whereas there is a unique class converging to a point not on  $H$ . We have a map  $\alpha: W \rightarrow M$  assigning to a filter its center and we endow  $W$  with the most economical topology making  $\alpha$  continuous. Then it looks easy to check that  $W$  is a bordered manifold. As the map  $\alpha$  is two-to-one above  $H$ , it gives a mapping  $\sigma: \partial W \rightarrow \partial W$  exchanging these two points. Now we have all the necessary ingredients to conclude as in the first step (1).

**Proof of the distinctive property in (5.5).**  $[\Rightarrow]$  (NB: this is the sense really needed for the corollary below). If  $H$  does not divide, then the bordered manifold  $W$  is connected, and so is a fortiori  $M_H$  which is obtained by identifying  $W$  with a replica  $W'$ . (Here and below, we use implicitly that the interior of  $W$  is naturally homeomorphic to  $M - H$ , and the general fact that a bordered manifold is connected iff its interior is.)

$[\Leftarrow]$  Assume that  $H$  divides  $M$ . Then  $W$  is disconnected, and then  $M_H$  is disconnected as follows from the construction. Indeed assume for (psychological) simplicity that  $W$  has two components  $W_+, W_-$ . Then  $M_H$  results from  $W \sqcup W' = (W_+ \sqcup W_-) \sqcup (W'_+ \sqcup W'_-)$  by attaching  $W_+$  with  $W'_-$  and  $W_-$  with  $W'_+$  and therefore  $M_H$  has two components. ■

This gives our sought-for:

**Corollary 5.7** A closed hypersurface  $H$  (as a point-set!) in a simply-connected manifold  $M$  divides the manifold  $M$ .

**Proof.** If  $H$  would not divide  $M$ , then the polarized covering  $M_H \rightarrow M$  is connected, violating the assumption that  $\pi_1(M) = 0$ . ■

In particular this implies what we really wanted in Prop. 5.1(d)

**Corollary 5.8** *Any leaf of a simply-connected surface divides.*

**Proof.** Point (b) of (5.1) implies that the leaf is a hypersurface in the sense of (5.3), whereas point (c) of the same (5.1) ensures that the leaf is a closed as a point-set. Thus we conclude with (5.7). ■

### 5.3 More analogies and divergences from Haefliger-Reeb

Albeit we shall not use it, we can push-forward the analogy with Haefliger-Reeb's theory. If  $\mathcal{F}$  is a foliation on a simply-connected surface  $S$ , then

(A) the leaf-space  $S/\mathcal{F}$  is still a 1-manifold (generally non-Hausdorff), cf. (5.9) below. In our setting the leaf-space needs not to be second-countable (equivalently Lindelöf, as manifolds are locally second-countable). So the leaf-space is a non-Hausdorff 1-manifold with possibly long “branches” or also with possibly uncountably many branches (consider e.g., the leaf-space of  $\mathbb{L}^2$  slitted along the closed set  $\mathbb{L}_{\geq 0} \times \omega_1$  and foliated vertically or the Prüfer type example depicted on Figure 2).

(B) However there is a little divergence with the metric case, for now the projection  $S \rightarrow S/\mathcal{F}$  needs not to be a (locally trivial) fibration. Indeed it is enough to consider slitted long planes  $\mathbb{L}^2 - (\{0\} \times \mathbb{L}_{\geq 0})$  foliated vertically to see that the leaf-type can jump erratically between the three open 1-manifolds (real-line, long ray and long line). Another perverse example is provided by the vertical foliation of the Moore surface (cf. Figure 2), where there is no jump in the topological type of the leaves, yet the projection  $M \rightarrow M/\mathcal{F}$  is not a fibration. If it would then since the base is  $\mathbb{R}$  which is contractible the fibration would be trivial (Feldbau-Ehresmann-Steenrod), and so the total space would be  $\mathbb{R}^2$  violating the non-metric nature of the Moore surface  $M$ .

As in Haefliger-Reeb [18, p. 122] the fact that a leaf appears at most once in a foliated chart (Prop. 5.1(b)) implies the:

**Corollary 5.9** *The leaf-space  $V = S/\mathcal{F}$  of a foliated simply-connected surface  $S$  is a one-dimensional manifold (generally non-Hausdorff), which is simply-connected (i.e.,  $\pi_1(V)$  is trivial, or equivalently  $V$  is divided by any puncture).*

**Proof.** (Just a translation of Haefliger-Reeb's argument.) To show that  $V = S/\mathcal{F}$  is a 1-manifold, it is enough to check that any point  $z \in V$  admits an open neighborhood homeomorphic to the number-line  $\mathbb{R}$ . Let  $\pi: S \rightarrow V$  the canonical projection (associated to the equivalence relation  $\rho$  of appurtenance to the same leaf); the leaf  $\pi^{-1}(z)$  meets at least one foliated chart  $O_i$ . The equivalence relation induced by  $\rho$  on  $O_i$  is, by Prop. 5.1(b), the relation  $\rho_i$  corresponding to the partition in parallel lines. Thus  $\pi(O_i)$  which is an open neighbourhood of  $z$  (since  $\rho$  is an open equivalence relation<sup>11</sup>), is homeomorphic to  $O_i/\rho_i$ , that is to the numerical line  $\mathbb{R}$ .

Regarding the second assertion (simple-connectivity of the leaf-space) we again follow Haefliger-Reeb. The complement of each leaf  $L$  (a closed subset of  $S$ ) has two components (Prop. 5.1(c)(d)); hence the complement of any point of  $V$  has also two components. This is equivalent to the simple-connectivity of  $V$  (compare lemma p. 113 in Haefliger-Reeb [18] which is a special case of (5.5), or formulate an appropriate exercise in algebraic topology using Seifert-van Kampen, or Mayer-Vietoris). ■

### 5.4 Hausdorffness of the leaf-space in the $\omega$ -bounded case

By the preceding section, the leaf-space of a foliated simply-connected surface is a 1-manifold. In the metric case, the non-Hausdorffness of the quotient is

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<sup>11</sup>This means that the saturation of any open set is open, or what amounts to the same that the canonical projection is open.

mostly catalyzed by Reeb components. Heuristically it is rather evident that there is no long Reeb components. More precisely if one assumes that there is a long transversal, then it is easy to deduce a continuous map from the long ray  $\mathbb{L}_+$  to the reals  $\mathbb{R}$  which is not eventually constant (by looking how the leaves emanating from a point on the transversal intercept a cross-section of a foliated chart). This gives some weight to the:

**Conjecture 5.10** *The leaf-space of any foliated simply-connected  $\omega$ -bounded surface is Hausdorff (which is probably always the long-line).*

Here is an outline of the difficulty appearing in an attempt of proof. Given two leaves  $L_1, L_2$  one would like to separate them. Of course if there is a leaf  $L$  which divides  $L_1$  from  $L_2$  in the sense of “Jordan” that is the  $L_i$  belong to two distinct components of  $M - L$ , then those components (projected in the leaf-space) will separate  $L_1$  from  $L_2$  in the sense of Hausdorff, and we are finished. Now we would like to show that under the  $\omega$ -boundedness condition, there is such a leaf  $L$ .

Probably more is true. Recall that a divisor is a (locally flat) hypersurface which is closed as a point set. In a simply-connected foliated surface any leaf is a divisor which is not a circle (5.1). Let us call *pseudo-line* a connected divisor in a simply-connected surface (say an *absolute*, for short) which is not the circle. (It can be the real-line, the long ray or the long line). Since any divisor in a simply-connected  $M^2$  divides (5.7), given 3 pseudo-lines in an absolute, either one of them divides the two others or no lines separates the remaining two. Call the first configuration *parallel*, and the second an *amoeba*. In the latter case the 3 pseudo-lines bound a bordered subregion namely the triple intersection of those halves of the  $L_i$  containing the remaining two pseudo-lines  $L_j, L_k$  ( $j, k \neq i$ ). Then we have the following strengthening of the conjecture:

**Conjecture 5.11** *Any 3 leaves of an  $\omega$ -bounded foliated absolute are parallel.*

Here is a somewhat more theological argument supporting this conjecture, which is perhaps not the most elementary, yet adumbrating a broader perspective. If not, then the three lines  $L_1, L_2, L_3$  are in the configuration of an amoeba. Then one can double the “amoeba” domain bounding the three curves  $L_i$  to get a sort of long pant. It is easy to show that the latter pant is  $\omega$ -bounded and of Euler characteristic  $-1$  (for instance with Mayer-Vietoris or by using the fact that the characteristic of a bagpipe is equal to that of the bag, cf. [15, Lemma 4.4]) Then conclude with the following conjecture (5.12) which has probably some independent interest (to be compared to the hairiness note [15] for flows).

## 5.5 Missing Euler obstruction

**Conjecture 5.12** *An  $\omega$ -bounded surface with negative Euler characteristic  $\chi < 0$  cannot be foliated.*

This is an intriguing version of the Euler-Poincaré obstruction. We think by experience that it must be true, yet the proof looks more involved than in the flow case (where the hypothesis was slightly different, namely non-zero  $\chi$ , cf. [15]). The example of  $\mathbb{L}^2$  with  $\chi = 1$  shows that the condition  $\chi \neq 0$  is not enough to obstruct foliability.

## 6 Poincaré-Bendixson arguments

In this section, we derive from Poincaré-Bendixson’s trapping argument under dichotomy (alias Jordan separation), several *universal* obstructions to transitivity not confined to the metric case. The complexity (of the proofs) raises with the topology quantified by the rank of the  $\pi_1$ . The method is basically a reduction to the dichotomic case by passing to double covers, with Poincaré-Bendixson’s method acquiring more punch when combined with Riemann’s branched covers.



When the total space fails to be dichotomic, some deeper versions of Poincaré-Bendixson (like those of Kneser, Markley, etc.) describing the dynamics on the Klein bottle enter into the arena. Ultimately we derive an almost complete classification of finitely-connected metric surfaces which are transitively-foliated. Besides, intransitivity transfers non-metrically, being conserved to any non-metric degeneracy of a finitely-connected metric surface provided its invariants (Euler character, ends-number and indicatrix) are kept unaltered. The soul concept formalizes this idea while unifying all results under a single perspective.

### 6.1 Dynamics on the bottle (Kneser, Peixoto, Markley, Aranson, Gutiérrez)

Beside the basic Poincaré-Bendixson obstruction, we require several other classic theorems describing the dynamics on the Klein bottle. Those rely on some magic arguments close to the Poincaré-Bendixson trapping, yet deviating from it inasmuch as they exploit a global cross-section.

**Lemma 6.1** (*Kneser 1924 [28]*) *Any foliated Klein bottle has a circle leaf.*

**Corollary 6.2** *The Klein bottle  $\mathbb{K}$  is foliated-intransitive.*

**Proof.** By Kneser (6.1) there is a circle leaf  $K$ . If it divides the bottle  $\mathbb{K}$  we are finished. Else cut the surface along  $K$  to get a *connected* compact bordered surface with  $\chi = 0$  and either one or two contours. By classification (2.4) these are resp. a (compact) Möbius band or an annulus. Deleting the boundary gives in both cases surfaces with  $\pi_1 \approx \mathbb{Z}$ , and conclude with (6.6) below. ■

**Lemma 6.3** (*Markley 1969, Aranson 1969, Gutiérrez 1977*) *The Klein bottle  $\mathbb{K}$  is flow-intransitive.*

**Proof.** The intransitivity of  $\mathbb{K}$  was first established by Markley 1969 [30] (independently Aranson 1969), yet the argument of Gutiérrez 1978 [16, Thm 2, p. 314–5] seems to be the ultimate simplification. We recall it for completeness.

By a lemma of Peixoto there is a global cross-section  $C$  to the flow (transverse circle). This circle is two-sided (its tubular neighborhood being oriented by the flow-lines is an annulus not a Möbius band). Also  $C$  is not dividing (a separation impeding transitivity). Cutting  $\mathbb{K}$  along  $C$  yields a connected bordered surface  $W$  with 2 contours with  $\chi$  unchanged equal to 0. By classification (2.4),  $W$  is an annulus. Orient its 2 contours  $C_1, C_2$  as the boundary of  $W$ , and the original surface is recovered by an orientation-preserving homeomorphism  $h: C_1 \rightarrow C_2$  (which we may assume, in reference to a planar model say  $W = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$ , to be a reflection about the vertical axis on  $C_1$  followed by a radial map  $C_1 \rightarrow C_2$ ). We denote  $h(p) = p'$ , just by a prime.

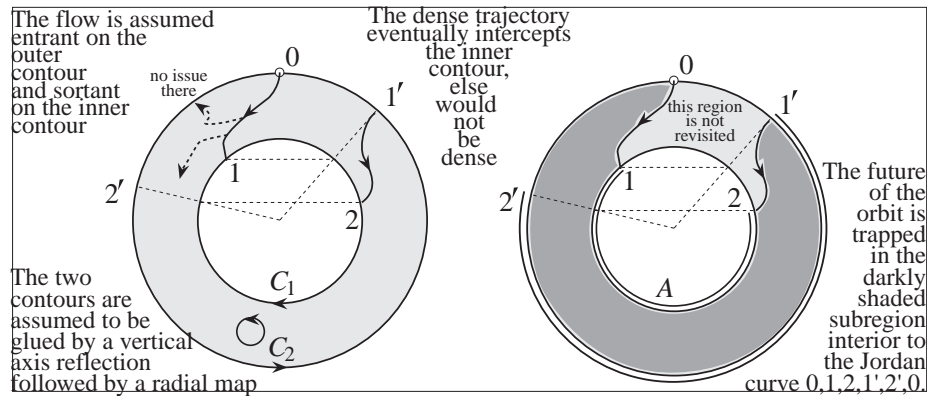


Figure 7: Flow intransitivity of the Klein bottle (Gutiérrez's proof)

Assume the flow entrant on the outer contour  $C_2$  and sortant on the inner contour  $C_1$ . A dense orbit must cross  $C$ , and w.l.o.g. we may suppose that the

forward-orbit is dense. Let 0 be a point on  $C_2$  whose forward-orbit is dense in  $\mathbb{K}$ . Since the inner contour  $C_1$  has a foliated collar where the flow is sortant, the dense orbit must eventually reach this collar and so intercepts  $C_1$  at some point, say 1. Then reflect vertically 1 and map it radially to get  $1' \in C_2$ . As before (denseness) the subsequent trajectory must again intercept  $C_1$ , at some position say 2. Consider  $h(2) = 2'$ , and notice that the subsequent orbit is trapped inside the dark subregion of Figure 7, violating denseness. Indeed, the future of  $2'$  will be an interception with the arc  $A = \overline{1, 2} \subset C_1$  determined such that the Jordan circuit  $0, 1, A, 2, 1', 2', 0$  [=flowing forwardly from 0 to 1, then following the arc  $A$ , next flowing backwardly from 2 to  $1'$  and finally moving injectively on the circle along the orientation specified by the triple  $1', 2', 0$ ] is null-homotopic in the annulus  $W$ , so bounds a disc  $D$  in  $W$ , which is the required trapping region, since  $h(A) \subset D$ . ■

## 6.2 Dichotomy obstructs oriented transitivity

The cornerstone is an oriented foliated avatar of Poincaré-Bendixson:

**Lemma 6.4** *An oriented foliation on a dichotomic surface has no dense leaf. Further an addendum is that no finite collection of leaves can be dense.*

**Proof.** This is the trapping argument of Poincaré-Bendixson, best understood by drawing a figure:

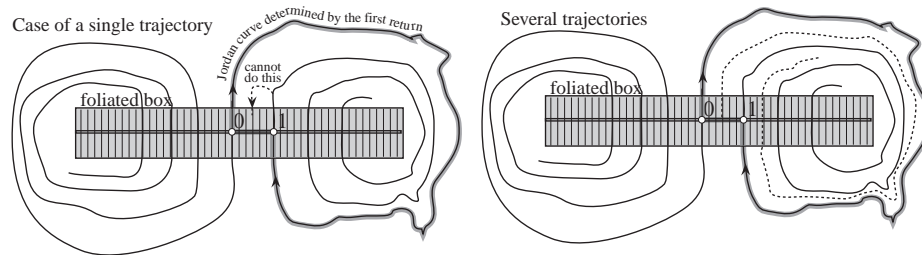


Figure 8: Foliated Poincaré-Bendixson argument

Assume that  $L$  is a dense leaf. Choose on it a point (called 0) and about 0 a foliated box  $B$ . Since  $L$  is dense it must reappear in the box  $B$ . W.l.o.g. assume this to be a forward interception w.r.t. the orientation (else reverse it). Call the first return to the cross-section 1. The piece of leaf from 0 to 1 closed by the cross-sectional arc from 1 back to 0 is a Jordan curve  $J$ . By dichotomy  $J$  divides the surface, trapping the future of the trajectory. In particular the next return to the cross-section, call it 2, occurs to the right of 1. By induction it follows that the successive returns occur in an order preserving fashion. On that picture one can safely superpose several trajectories (=oriented leaves) so as to deduce the addendum. ■

**Remark 6.5** Lemma 6.4 reproves that a 1-connected surface lacks dense leaves (a foliation on a 1-connected manifold being automatically orientable (4.1)).

## 6.3 Foliated surfaces with infinite cyclic group

Our initial intention was to apply the Haefliger-Reeb theory to the following proposition, by passing to the universal cover while arguing that the lifts of the dense leaf cannot be dense. (We were not able to present a decent proof and crudely put, met some difficulties in showing that the generating deck-translation acts as translations or gliding reflections of  $\mathbb{R}^2$ .)

**Proposition 6.6** *A (non-metric) surface whose fundamental group is infinite cyclic lacks a transitive foliation (i.e., with at least one dense leaf).*

**Proof.** If orientable, the surface is dichotomic (2.15) and we conclude with (6.4) after orienting the foliation up to passing to the double cover (4.1). As the group is  $\mathbb{Z}$ , it stays so under finite covering, which also preserves orientability. If not orientable, the surface constructs its orientation double cover (2.9) and we are reduced to the previous case. ■

## 6.4 Free groups of rank two under dichotomy

Recall as a motivation Dubois-Violette's transitive (indeed minimal) foliation on the thrice punctured plane with fundamental group  $F_3$  (free on three letters). Also the Kronecker torus punctured once shows a minimal foliation on a surface with  $\pi_1 \approx F_2$  (free of rank 2). The following result (surely well-known in the metric case, albeit we did not checked the literature carefully) shows the sharpness of those examples as having the minimum complexity for the fundamental group permitting a dense leaf. In particular 3 is the minimal number of punctures required to the plane to manufacture a *labyrinth* (=foliation with a dense leaf).

**Proposition 6.7** *A dichotomic surface with fundamental group  $\pi_1$  free of rank 2 is foliated-intransitive.*

This applies to the twice-punctured Moore surface, and more generally to any twice-punctured simply-connected surface, in view of (2.23).

**Proof.** If the foliation is orientable, then the foliated version of Poincaré-Bendixson (6.4) concludes. Otherwise, pass to the 2-fold orienting cover (4.1), which by the branched covering argument of (2.17) is still dichotomic, reducing again to the Poincaré-Bendixson obstruction (6.4). ■

## 6.5 Free groups of rank two (non-orientable cases)

The above (6.7) does not apply to  $\mathfrak{M}_*$  (punctured Möbius band), which is  $\mathbb{R}P_{**}^2$  (twice punctured projective plane), which has  $\pi_1 \approx F_2$  (2.23), but not dichotomic by (2.14). Yet the method of branched covers still applies:

**Proposition 6.8** *The twice-punctured projective plane  $\mathbb{R}P_{**}^2 = \mathfrak{M}_*$  is foliated-intransitive.*

**Proof.** *Orientable case.* If the foliation is orientable, take a compatible flow on  $\mathbb{R}P_{**}^2$  (4.2). By a standard method à la Beck (4.5) the flow extends to  $\mathbb{R}P^2$ . Passing to the universal cover  $S^2$ , Poincaré-Bendixson obstructs transitivity.

*Non-orientable case.* If not, the foliation determines a double orienting cover  $p: \Sigma \rightarrow \mathbb{R}P_{**}^2$  (4.1). Looking around the punctures we can by Riemann's trick (2.16) compactify this map to a branched covering  $\Sigma^* \rightarrow \mathbb{R}P^2$  (ramified at  $R$  a sublocus of the punctures). Thus

$$\chi(\Sigma^*) = 2\chi(\mathbb{R}P^2) - \deg(R).$$

Since  $\deg(R) \leq 2$ ,  $\chi(\Sigma^*) \geq 0$ . Since  $\Sigma^*$  is connected we have also  $\chi(\Sigma^*) \leq 2$ .

If  $\chi(\Sigma^*) = 2$ , then  $\Sigma^* \approx S^2$  and Poincaré-Bendixson concludes.

If  $\chi(\Sigma^*) = 1$ , we have  $\mathbb{R}P^2$  and argue as above (orientable case).

If  $\chi(\Sigma^*) = 0$ , then we have either the Klein bottle  $\mathbb{K}$  or the torus  $\mathbb{T}^2$ . In the first case, lift the foliation to  $\Sigma$ , take a compatible flow (4.2) and “extend” it to  $\Sigma^*$  (4.5), violating the flow-intransitivity of Klein  $\mathbb{K}$  (6.3).

The toric case requires a separate argument. We have the branched covering  $\mathbb{T}^2 = \Sigma^* \rightarrow \mathbb{R}P^2$  ramified at two places, as  $\deg(R) = 2$ . Exchanging sheets induces an involution  $\sigma$  of the torus which is orientation reversing (the quotient being non-orientable) with two fixed points. One obstruction to this is geometric amounting essentially to the Klein-Weichold (1876–1883) classification of orientation reversing involutions on oriented closed surfaces (relevant to the

real algebraic geometry of curves). In fact we merely need the very basic fact, asserting that the linearization in the small of such an involution near a fixed point is a symmetry about a line, violating the isolated nature of the above fixed points. (Formal proof of this in the topological case came slightly later with the era of Schoenflies, Brouwer, Kerékjártó.) ■

Beside  $\mathbb{R}P_{**}^2 = \mathfrak{M}_*$ , another specimen with  $\pi_1 = F_2$  is  $\mathbb{K}_*$  (punctured *Klein bottle*, i.e. non-orientable closed surface with  $\chi = 0$ , so  $\mathbb{K} = S_{2c}$  is also the sphere with 2 cross-caps). Again we expect a similar result:

**Lemma 6.9** *The punctured Klein bottle  $\mathbb{K}_*$  is foliated-intransitive.*

**Proof.** We break the argument in two parts; the first being a repetition of the method employed so far, which in the present case seems sterile:

**The usual method foils.** If the foliation is orientable we are done by the flow-intransitivity of Klein  $\mathbb{K}$  (6.3). If not the foliation defines its oriented 2-fold covering  $\Sigma \rightarrow \mathbb{K}_*$  (4.1), which we compactify into a branched covering  $\Sigma^* \rightarrow \mathbb{K}$ . In particular:

$$\chi(\Sigma^*) = 2\chi(\mathbb{K}) - \deg(R).$$

As  $0 \leq \deg(R) \leq 1$ , it follows  $-1 \leq \chi(\Sigma^*) \leq 0$ . Hence  $\Sigma^*$  is either  $\mathbb{K}$  or  $T^2$  if  $\chi = 0$  or  $N_3$  the sphere with 3 cross-caps when  $\chi = -1$ .

The case of  $\mathbb{K}$  is easily ruled out by Markley's intransitivity (6.3).

The other cases are harder. In the toric case  $\deg(R) = 0$ , so that the compactifying covering is unramified. This means that about the puncture the foliation is oriented, etc.

**A new trick is required.** Maybe a more efficient argument is to use the index formula for line-fields (Poincaré, Bendixson, Kerékjártó, Hopf, etc.). Since there is a unique singularity, the index at the puncture is zero. Thus the foliation extends to  $\mathbb{K}$ . (If the singularity looks like a letter “X” with opposite hyperbolic sectors and opposite focus-type sectors with leaves converging to the puncture, then there is no such extension! Yet such a scenario impedes transitivity due to the focusing sectors.) Once the foliation is extended, conclude with Kneser 1924 (6.1) or rather its corollary (6.2). ■

Uniting the forces of the two previous propositions we deduce:

**Lemma 6.10** *A non-orientable metric surface with  $\pi_1 = F_2$  is intransitive.*

**Proof.** Such a surface is homeomorphic either to  $\mathbb{R}P_{**}^2$  or  $\mathbb{K}_*$  by (2.30). ■

The ultimate generality is to relax the metric proviso:

**Theorem 6.11** *A non-orientable surface with  $\pi_1 = F_2$  is intransitive.*

**Proof.** The idea is to reduce to the metric case via an appropriate exhaustion. If  $L$  is a dense leaf then  $L$  is either  $\mathbb{R}$  or the long-ray  $\mathbb{L}_+$  (cf. (7.3) below). Up to deleting the long-side of  $L$  (which does not affect the assumption made on our surface  $M$  by (2.25)) we may assume that  $L$  is the real-line. Choose now a  $\pi_1$ -calibrated exhaustion (2.21)  $M = \bigcup_{\alpha < \omega_1} M_\alpha$  by Lindelöf subregions with  $\pi_1(M_\alpha) \approx \pi_1(M) = F_2$ . Since  $L$  is Lindelöf, there is  $\beta < \omega_1$  such that  $M_\beta \supset L$ ; and  $L$  being dense in  $M$  it is a fortiori so in  $M_\beta$ . Since  $M$  is non-orientable, it contains a one-sided Jordan curve  $J$  (2.11) (whose tubular neighbourhood  $T$  is a Möbius band). Since  $T \supset J$  is Lindelöf we may assume that  $M_\beta \supset T$  as well, violating the metric case (6.10) of the theorem. ■

This applies to the Moore surface  $M$  with two cross caps, denoted  $M_{2c}$ , as well as to  $M_{*,c}$  the Moore surface punctured once and cross-capped once (apply (2.24) and (2.23)). The next section studies the sharpness of those results, while giving some sporadic extensions to groups of rank 3.

## 6.6 The monolith of finitely-connected metric surfaces

This section aims to classify those finitely-connected metric surfaces which are transitive (and those which are not). In the subsequent section we deduce non-metrical transfers of intransitivity.

In view of Kerékjártó (2.27) any finitely-connected metric surface is homeomorphic to a finitely-punctured closed surface. Hence by the Möbius *et al.* classification (2.4) any such surface derives from the sphere  $S := S^2$  through iteration of the three operations (1) handle surgery, (2) cross-capping (3) puncturing. Thus we can tabulate a “monolith” for all such surfaces (Figure 9 below), where right-arrows are *puncturing* (denoted by a starred subscripts “ $*$ ”, e.g.  $S_* = \mathbb{R}^2$ ), up double-arrows are *handle attachments* ( $\Sigma_g$  denoting the orientable closed surface of genus  $g$ ), and left-squig-arrows are *cross-caps* (denoted by subscripts “ $c$ ”). Boldface fonts denote the rank of the (fundamental) group when it is free. (Given a rank there are only finitely many metric surfaces with the prescribed group.) The exotic arrows (not fitting with the hexagonal lattice) arise from the well-known relations in the monoid of closed surfaces (under-connected sum) inherent to the classification theorem (2.4) in term of  $\chi$ , and the orientability character. (For instance attaching a handle to a non-orientable surface amounts to 2 cross-caps, both decreasing  $\chi$  by 2 units.)

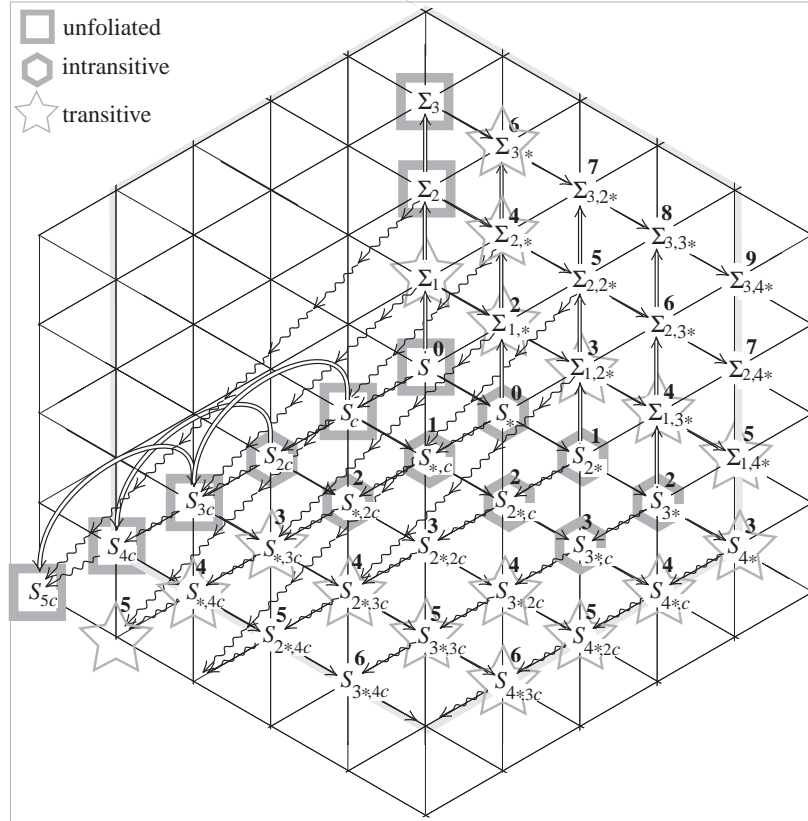


Figure 9: The monolith of finitely-connected metric surfaces

*Squares* indicate those surfaces which cannot be foliated (Euler-Poincaré obstruction  $\chi \neq 0$ ). *Hexagons* show surfaces which are foliated-intransitive in view of previously listed obstructions (6.4), (6.6), (6.7), (6.10) and (6.16) below. *Stars* show surfaces which are foliated-transitive (as discussed below).

**Lemma 6.12** *If a surface is transitive, then so is its punctured version (just puncture outside a dense leaf).*

Thus we need only to establish the transitivity of “minimal” models with respect to puncturing. For instance  $\Sigma_1$  the torus is transitive (Kronecker foliation) and this propagates right-down (on Figure 9).

## 6.7 Transitive examples via surgery (Peixoto, Blohin)

To construct transitive foliations, we can use a surgical device (due e.g., to Peixoto 1962 [36], Blohin 1972 [7]):

**Lemma 6.13** *The following surfaces are transitively foliated:*

- (1)  $\Sigma_{g,*}$  the once punctured orientable surface of genus  $g \geq 1$ ;
- (2)  $S_{*,gc}$  the once punctured sphere with  $g$  cross-caps  $g \geq 3$ .

**Proof.** Start with a (Kronecker) irrational foliation of the torus  $T^2$ . Pick two foliated boxes and apply the woodpecker-surgery (Fig. 10.a). Connecting by a handle the two contours, and deleting the arc (saddle connection) shows that  $\Sigma_{2,*}$  is transitive (the arc deletion amounts to a single puncturing as the handle is thought of as infinitesimal so that the two depicted arcs are in reality just one). For higher genres, consider an alignment of such flow-boxes (cf. Fig. 10.b), and delete the thick arc (3 pieces, but connected!) proving (1). Regarding (2) we cross-cap the contours (cf. Fig. 10.c) and delete the thick arc. Since the torus with one cross-cap is  $\approx S_{3c}$  this proves (2). ■

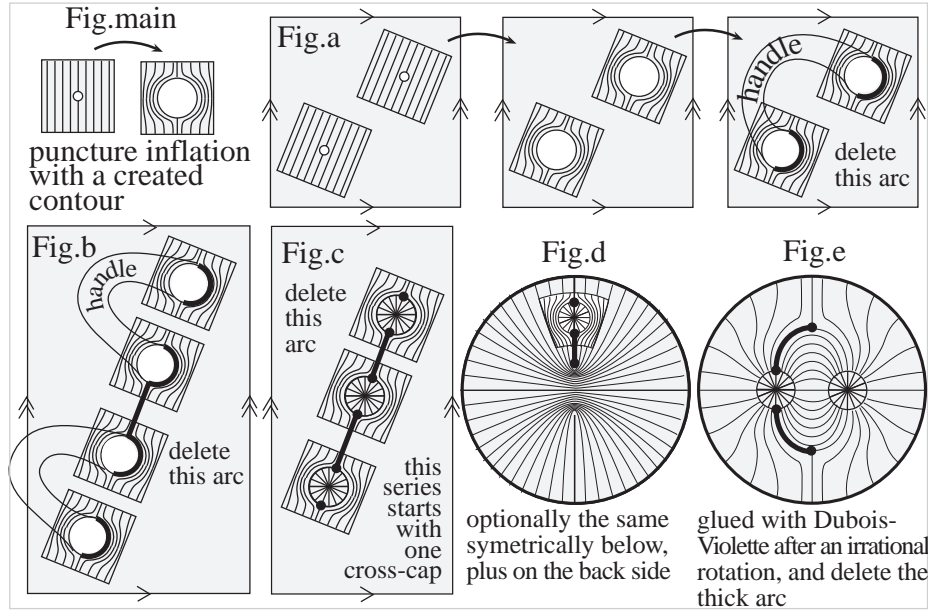


Figure 10: The woodpecker-surgery of Peixoto-Blohin

Doing the same surgery in the Dubois-Violette foliation (Fig. 10.d), shows:

**Lemma 6.14** *The sphere  $S_{n*,kc}$  with  $n$  punctures and  $k$  cross-caps is transitive for  $n = 4$  and  $0 \leq k \leq 4$ .*

Together with (6.13) (and keeping a view over the monolithic Figure 9), this gives a complete knowledge of which  $S_{n*,kc}$  are transitive, except when  $(n, k)$  takes the values  $(3, 2)$ ,  $(2, 2)$ ,  $(3, 1)$ .

The first case  $(3, 2)$  is transitive by gluing Fig. 10.e with Dubois-Violette's disc (Figure 1, Rosenberg's version). This piece of the puzzle suffices to establish in view of the combinatorics of Figure 9 the:

**Proposition 6.15** *An open metric surface of finite-connectivity (=rank of the  $\pi_1$ )  $\geq 4$  is transitive.*

## 6.8 Sporadic obstruction in rank 3

The last case  $(n, k) = (3, 1)$  (of the previous section) is intransitive by the following (using again Riemann's branched coverings conjointly with the Poincaré-Kérékjártó-Hopf index formula):

**Lemma 6.16** *The thrice-punctured projective plane  $S_{3*,c}$  is foliated-intransitive.*

**Proof.** If the foliation is orientable, then we are reduced to the flow-intransitivity of the projective plane  $\mathbb{R}P^2$  (which boils down to that of  $S^2$  prompted by Poincaré-Bendixson). Otherwise we construct the double cover  $\Sigma \rightarrow M := S_{3*,c}$  rendering the foliation oriented (4.1). Compactify this to a branched covering  $\Sigma^* \rightarrow M^* = \mathbb{R}P^2$  via Riemann's trick (2.16). By Riemann-Hurwitz  $\chi(\Sigma^*) = 2\chi(\mathbb{R}P^2) - \deg(R)$ . As there are 3 punctures,  $0 \leq \deg(R) \leq 3$ . Hence  $-1 \leq \chi(\Sigma^*) \leq 2$ . If  $\chi(\Sigma^*) = 2$ , then we have  $S^2$  or  $\mathbb{R}P^2$  both precluded by Poincaré-Bendixson. If  $\chi(\Sigma^*) = 0$ , then we have  $T^2$  or  $\mathbb{K}$ . The former is excluded since the sheet exchange involution must be orientation reversing hence cannot fix isolated points (Klein-Weichold argument already used in (6.8)). The Klein option  $\mathbb{K}$  is precluded by Markley's flow-intransitivity of the Klein bottle (6.3). Finally when  $\chi(\Sigma^*) = -1$ , we have  $\deg(R) = 3$ . This means by construction that all three punctures are non-orientably foliated. Thus they have each a semi-integral index. By the index formula they sum up to  $\chi(\mathbb{R}P^2) = 1$ , violating arithmetics modulo one  $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} \neq 1 = 0$ . ■

At this stage the only remaining bastion of resistance is the case  $S_{2*,2c}$  (twice punctured Klein bottle), of which it is not completely trivial to decide its transitivity issue. (We do not know yet the answer.) This is surely well known yet confess that presently we failed to present a decent proof.

## 6.9 Intransitivity transfer from the metric soul

As seen in (2.32) it is legitimate to think of any (non-metric) surface of finite-connectivity (i.e. with fundamental group of finite rank) has having a metric *soul* capturing the salient invariants  $(\chi = 1 - b_1, \varepsilon, a)$  of connectivity, the ends number and indicatrix (=orientable or not) which in the metric case constitutes a complete system of topological invariants (2.27). A foliated application of this soul-method (very akin to Nyikos' bagpipes) is the transfer of intransitivity from the metrical soul to the whole manifold:

**Proposition 6.17** *If the soul of a finitely-connected (non-metric) surface is intransitive, then so is the whole surface.*

**Proof.** The argument is similar to (6.11). Assume  $M$  transitive with dense leaf  $L$ . We know that  $L$  is either  $\mathbb{R}$  or the long-ray  $\mathbb{L}_+$  (cf. (7.3) below). Deleting from  $M$  the long-side of  $L$  does not change the invariants  $(\chi, \varepsilon, a)$  by virtue of (2.25), hence keeps invariant the soul type by (2.32). After this long slit, we have a new leaf  $L \approx \mathbb{R}$  which is still dense. As  $L$  is Lindelöf, it is contained in a Lindelöf subregion  $U \supset L$  (random amalgam of charts covering  $L$ ). By the kernel killing procedure (2.20) we can enlarge  $U$  into a soul  $S \supset U$  so that  $\pi_1(S) \rightarrow \pi_1(M)$  is isomorphic.  $L$  being dense in  $M$ , it is a fortiori in  $S$ , violating the soul intransitivity. ■

Applying (6.16) we find:

**Corollary 6.18** *(Non-metric) surfaces with  $\pi_1 = F_3$  and 3 ends are intransitive.*

This applies for instance to the Moorization (cf. e.g. [14]) of the bordered surface given by the sphere with 3 holes and 1 cross-cap (i.e. 3-holed  $\mathbb{R}P^2$ ). Notice that the full-theory of the soul is not truly required for such simple-minded Moore type surfaces whose geometric structure is sufficiently explicit so as to replace the soul by a calibrated exhaustion by subregions having the same topological type (just add successively the thorns).

## 6.10 Biminimal foliations (bidirectional denseness)

In this section we address a somewhat specialized question, involving methods of Bendixson, Kerékjártó and Mather. Albeit Dubois-Violette's foliation of

$S_{4*}^2$  (4 punctures in  $S^2$ ) is minimal, the 4 leaves converging to the punctures (separatrices) fails to be dense in one direction. In contrast, all leaves of the Kronecker irrational foliation of the torus are bidirectionally dense.

**Definition 6.19** A 1-foliation is *biminimal* if all leaves are bidirectionally dense, i.e. both semi-leaves emanating from any point are dense.

By the sequential-compactness argument (7.1) all leaves of a biminimal foliation are real-lines  $\mathbb{R}$  (provided ambient dimension  $\geq 2$ ). We may wonder if  $S_{4*}^2$  (fourth-punctured sphere) is biminimal. The negative answer is supplied by the following Bendixson alternative:

**Lemma 6.20** *In a foliated punctured plane  $\mathbb{R}_*^2$ , there is either a circle leaf enclosing the puncture or there is a leaf converging to the puncture. Moreover both alternatives exclude mutually if the first option occurs countably many times with a nested collection of circle leaves shrinking to the puncture.*

**Proof.** Mather 1982 [32, p.246, §7], refers to Bendixson original paper or to Kerékjártó 1923 [26, p.256]. Here is a brief outline of the argument as modernized by Birkhoff, Nemytskii-Stepanov, Reeb, etc. The classical Poincaré-Bendixson argument shows that dichotomy implies properness in the flow case, to which we may reduce as  $\pi_1 = \mathbb{Z}$  by passing to the double cover (4.1). Then a standard Zorn lemma argument creates a compact leaf, provided there is a leaf with compact semi-leaf closure. The compact circle leaf encloses the puncture, since otherwise it bounds a disc (recall that a proper power of the generator of  $\pi_1$  cannot be realised by a Jordan curve). ■

**Corollary 6.21** *Any punctured surface (metric or not) lacks a biminimal foliation.*

Since any finitely-connected open metric surface possesses an end neighbourhood homeomorphic to a punctured plane (2.27), we have the special case:

**Corollary 6.22** *A finitely-connected metric surface cannot be biminimally foliated, except the torus.*

**Proof.** If compact, the surface has  $\chi = 0$  (4.4), so is the torus or the Klein bottle  $\mathbb{K}$  (2.4). The latter option is precluded by Kneser 1924 (6.1).

In the open case, by Kerékjártó's end theorem (2.27), the surface is a punctured one and (6.21) concludes. ■

Those results fails to tell if the infinite connected sum of tori have a (bi)minimal foliation. Also which metric surfaces are biminimally foliated? Besides, does the last corollary extend to non-metric surfaces? It seems that the Lindelöf exhaustion trick does not work well. Note that the doubled Prüfer surface  $2P$  is not biminimal (indeed not even minimally foliated by (7.6)). Baillif's example in BGG2 [5] of a minimally foliated non-metric surface foliated by short leaves is not biminimal, raising the:

**Question 6.23** *Can we find a non-metric biminimally foliated surface?*

## 7 Gravitational effects (quantum radiation at the microscopic scale)

### 7.1 Long semi-leaves are tame

Given a point in a manifold foliated by curves, then after fixing one of the two possible directions there is a unique motion starting from the point prescribed by the foliation. Such a “trajectory” referred to as a *semi-leaf*, is a bordered 1-manifold (under the leaf-topology).



**Proposition 7.1** *Long semi-leaves in 1-foliated manifolds are properly embedded.*

**Proof.** By classification of bordered 1-manifolds (in 3 species:  $[0, 1]$ ,  $[0, \infty)$  and  $\mathbb{L}_{\geq 0} = [0, \omega_1)$  closed long ray) our non-metric semi-leaf is the closed long ray. The latter being sequentially-compact, we conclude with (7.2) below. ■

**Lemma 7.2** (i) *Let  $f: X \rightarrow Y$  be a continuous map, where  $X$  is sequentially-compact (sekt for short), and  $Y$  Hausdorff and first-countable. Then the mapping  $f$  is closed.*

(ii) *In particular  $f(X)$  is closed and if furthermore  $f$  is injective, then  $f: X \rightarrow f(X)$  is a homeomorphism onto its image.*

**Proof.** First recall that a closed subset of a space is sequentially-closed, and conversely if the space is first-countable.

Thus for (i), it is enough to show that  $f(F)$  is sequentially-closed, whenever  $F$  is closed. So let  $y_n \in f(F)$  be a sequence converging to  $y \in Y$ . Choose  $x_n \in F$  such that  $f(x_n) = y_n$ . Since  $X$  is sekt, we may extract a subsequence  $x_{n_k} \rightarrow x \in F$ , converging to  $x$ , say. By continuity  $f(x_{n_k}) \rightarrow f(x)$ . Since  $Y$  is Hausdorff, it follows  $y = f(x) \in f(F)$ . q.e.d.

(ii) The continuity of the inverse follows from  $f$  being a closed mapping. ■

## 7.2 Transitivity implies separability

When a manifold (indeed a space) has a transitive flow (one dense orbit) then the phase-space is separable (rational times of a dense orbit). If a manifold is transitively foliated (one dense leaf), then as the latter can be long it is not obvious that the ambient manifold has to be separable. It is even trivially false as exemplified by a long 1-manifold trivially foliated by itself. Ruling out this trivial exception we have:

**Proposition 7.3** *A dense leaf of a 1-foliation on a manifold  $M^n$  of dimensionality  $n \geq 2$  is homeomorphic (w.r.t. the leaf topology) to the real-line  $\mathbb{R}$  or the long ray  $\mathbb{L}_+$ . Furthermore  $M^n$  is separable.*

**Proof.** By classification of 1-manifolds the leaf belongs to one of the following type: circle  $\mathbb{S}^1$ , real-line  $\mathbb{R}$ , long ray  $\mathbb{L}_+$  and long line  $\mathbb{L}$ . The two extreme items in this list are sequentially-compact, thus always embedded and closed as point-sets by (7.2). Thus by (the elementary case of) invariance of dimension (Brouwer *et al.* in general, yet not required presently), the dense leaf  $L$  cannot be of those two types as  $n \geq 2$ , whence the first assertion. Regarding the separability clause, it is plain when  $L \approx \mathbb{R}$ . Assuming  $L \approx \mathbb{L}_+$ , we may at any point  $p$  of  $L$  split the leaf in a short  $L_{\leq p}$  and a long  $L_{\geq p}$  semi-leaf, resp. homeomorphic to  $\mathbb{R}_{\geq 0}$  and  $\mathbb{L}_{\geq 0}$  (closed long ray). The latter being sequentially-compact, it fails to be dense. Thus only the short semi-leaf can contribute to denseness, and separability of  $M$  follows. ■

**Remark 7.4** For foliations of higher dimensionality, the above (7.3) fails drastically. An example of Martin Kneser shows how to foliate with a *unique* surface-leaf a non-metric 3-manifold of the Prüfer type (cf. for a picture, [4], arXiv version). Kneser's example can easily be 'stretched' so as to render it non-separable.

**Corollary 7.5** *An  $\omega$ -bounded surface is intransitive, except if it is the torus.*

**Proof.** If transitively foliated, the surface is separable (7.3). Being also  $\omega$ -bounded it is compact. Since it is foliated,  $\chi = 0$  (4.4). So by classification (2.4), it is either the torus or the Klein bottle, which is intransitive (6.2). ■

### 7.3 Miniature black holes (Prüfer, R. L. Moore, Baillif)

This and the subsequent section exemplify a geometric obstruction to foliated-transitivity lying beyond the algebraic obstruction encoded in the fundamental group or better in the soul (6.17), as well as beyond the point-set obstruction of non-separability (7.3). Thus the present obstruction (due to M. Baillif) is the first (within the scope of this paper) being truly non-metrical, yet still of a geometric nature prompted by the granularity of particular non-metric manifolds, namely those of the Prüfer type (including the Moore and Calabi-Rosenlicht surfaces, plus of course many other specimens having a similar morphology). Using the phase-transition metaphor, this amounts to a volatile-gaseous (=transitive) configuration which embedded into the non-metrical fridge becomes frozen-intransitive.

We already know that Cantor's long ray is responsible of some black hole phenomenology at the macroscopic scale [4]. For instance the long cylinder  $S^1 \times \mathbb{L}_+$  imposes to each foliated structure to be either ultimately vertical (foliated by straight long rays) or asymptotically horizontal (with slices  $S^1 \times \{\alpha\}$  occurring as leaf for a *closed unbounded (club)* subset of  $\alpha$ 's running in the long-ray factor). So one can essentially imagine a super-massive black hole hiddenly sitting at the long end of the cylinder and dictating the destiny of any foliated structure, thought of as an *ether* ( $\approx$  *substrat physico-chimique* in R. Thom's jargon) evidencing the gravitational features of the manifold. We are dealing here with a purely naked-topological form of gravitation without metric (and the allied Riemann curvature tensor), yet still reasonably qualifiable as "geometric".

Apart from Cantor's long ray (of dimension 1) the other charismatic prototype of non-metric manifold (requiring two dimensions) is the Prüfer construction. Especially we have,  $P$ , the bordered Prüfer surface (constructed by a aggregating rays to an open half-plane, very akin to projective geometry, esp. the blow-up operation) which—by folding the contours—produces the Moore surface. This can be thought of as an upper half-plane with many (infinitesimal) 'teats' hanging down the boundary (horizontal line), compare Figure 2 for a poor depiction. Clearly something 'erotic' must happen near the 'boundary' (or rather what remains thereof—that is nothing!), and by analogy with Cantor's super-massive black hole scenario, we now imagine a 'continuous' series of nano-black holes materialised by the folded contours of the Prüfer surface  $P$ , called the *thorns* of the Moorization. (If you prefer imagine the little black holes located at each teat's extremity.) The latter effects a quantum radiation at the microscopic scale as shown by the following technique of M. Baillif ( $\approx$  2008–9):

**Theorem 7.6** *In any foliation of the Moore surface, almost all (=all but countably many exceptions) thorns are semi-leaves. Similarly, in any foliation of the Calabi-Rosenlicht surface (=doubled Prüfer surface  $2P$ ) almost all bridges are leaves. (Here Bridges refer to the contours of  $\partial P$  viewed in  $2P$ .)*

**Proof.** Let  $P \rightarrow M$  be the contour-folding projection from Prüfer-to-Moore. The boundary  $\partial P$  decomposes as a continuum of real-lines contours, whose respective image are the thorns denoted  $T_x$  (homeomorphic to a semi-line  $\mathbb{R}_{\geq 0}$ ) indexed by  $x \in \mathbb{R}$ . The complement of all thorns is called the *core* (of the Moore surface). The core  $U$  being a cell  $\approx \mathbb{R}^2$ , hence separable, let us fix  $D \subset U$  a countable dense set. Let  $\mathcal{F}_D$  be the set of leaves of the foliation (denoted  $\mathcal{F}$ ) passing through the points of  $D$ .

If the thorn  $T_x$  is not a semi-leaf of  $\mathcal{F}$ , then there is a point  $y \in T_x$  such that  $L_y$  (=the leaf through  $y$ ) deviates into the core. Then we can find nearby a leaf  $L$  in the collection  $\mathcal{F}_D$  intercepting  $T_x$  (Fig. 11.a).

Choosing for each such  $x$  an intercepting leaf defines a map

$$\varphi: \text{White thorns} := \{x \in \mathbb{R} : T_x \text{ is not a semi-leaf}\} \longrightarrow \mathcal{F}_D.$$

Now a leaf can intersect at most countably many thorns. This follows from the squatness of the Moore surface  $M$  (i.e. any continuous map from the long ray to  $M$  is eventually constant) and the Prüfer topology which shows that

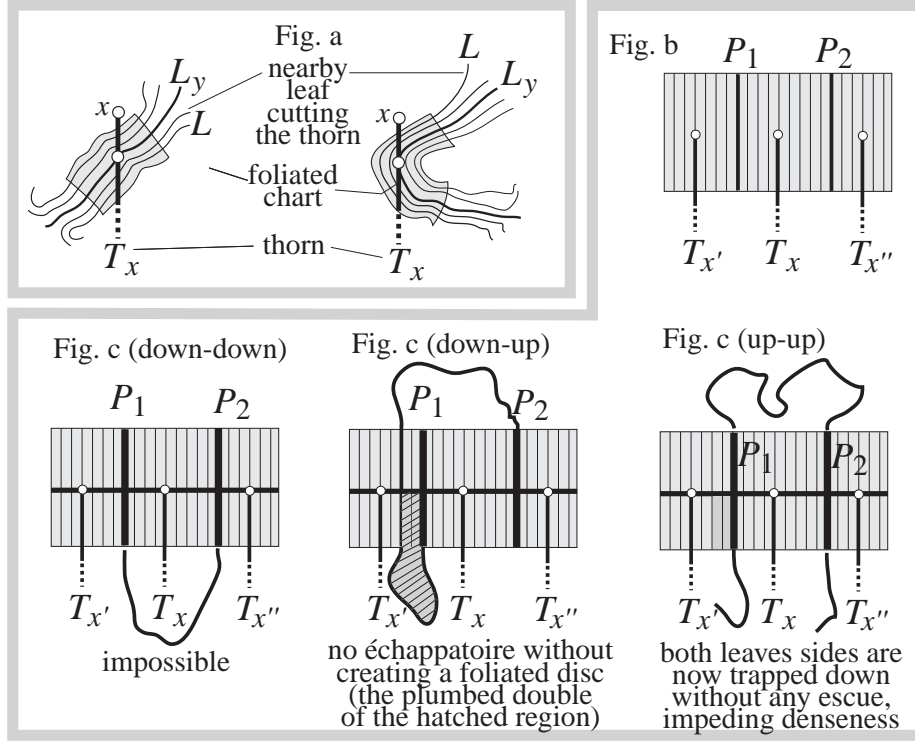


Figure 11: Interception of thorns by leaves and black thorns sandwiches

any subset of the Moore surface having at most one point on each thorn is discrete. Therefore long leaves are precluded and short ones cannot contain an uncountable discrete subset.

Hence the map  $\varphi$  is at most  $\omega$ -to-1 (denumerable fibres), whence the countability of the set of “White thorns”. The argument for the doubled Prüfer is nearly identical, hence left as an exercise. ■

#### 7.4 Intransitivity of the thrice-punctured Moore surface

The Moore surface  $M$  itself, being simply-connected (Seifert-van Kampen applied to the canonic open cover by individual thorns added to the core), is intransitive, as deduced either via Schoenflies (5.1) or Poincaré-Bendixson (6.4).

After one or two punctures there is an universal obstruction dictated by the fundamental group (6.6) resp. (6.7). In the Moore case, the argument can be done by hand:

**Optional easy argument by hand.** One adds successively the thorns of  $N = M_{n*}$  the Moore surface with  $0 \leq n \leq 2$  punctures: i.e.,  $N = (\text{core } U) \cup \bigcup_{x \in \mathbb{R}} T_x$  which is covered by the  $M_x := U \cup T_x$ . Assuming  $M$  transitive with dense leaf  $L$  we know that  $L$  is short (squatness of Moore). Thus  $L$  is Lindelöf and therefore contained in  $M_D := \bigcup_{x \in D} M_x$  a countable union of  $M_x$  ( $D \subset \mathbb{R}$  denumerable). Using either Morton Brown’s monotone union theorem (or the classification of metric 1-connected surfaces (2.5) or even better a home-made homeomorphism) it is easy to see that  $M_D$  is homeomorphic to the core  $U$ . Since  $U$  contains  $L$  densely, this violates the intransitivity of the core.

Doing more than  $\geq 3$  punctures in Moore there is no universal algebraic obstruction (recall Dubois-Violette), but a gravitational one (remind Baillif):

**Proposition 7.7** *The Moore surface with  $n \geq 3$  (hence all  $n$ ) punctures  $M_{n*}$  is foliated-intransitive.*

**Proof.** (with some little bluff, but hopefully convincing enough!) Truncate the punctured surface  $N := M_{n*}$  (foliated by  $\mathcal{F}$ ) by looking at the subregion  $U$  lying below a line above which all punctures are lying, and which therefore is homeomorphic to the Moore surface  $M$ . By (7.6) applied to  $(U, \mathcal{F}_U)$  almost

all thorns are “black”, i.e. semi-leaves of the foliation. Fix any black thorn  $T_x$ , and about its extremity  $\partial T_x$  a foliated chart  $B$ . Assuming that  $L$  is a dense leaf of  $\mathcal{F}$  it will certainly appear on both sides of  $T_x$  regarded in the foliated-box  $B$  as 2 plaques  $P_1, P_2$  separated by the plaque  $P$  of  $B$  containing  $T_x$ . Since the set of white thorns is countable, its complementary set of black thorns is dense (Baire). Thus we can squeeze in sandwich the 2 plaques  $P_1, P_2$  by 2 new black thorns  $T_{x'}$  and  $T_{x''}$  (Fig. 11.b) such that within the box  $B$ ,  $P_1$  separates  $T_{x'}$  from  $T_x$  and  $P_2$  separates  $T_x$  from  $T_{x''}$ . As the two  $P_i$  belong to the same leaf they must be somehow connected.

Agreeing that the box is vertically foliated, both plaques  $P_i$  will be either connected through their bottoms (down-down), in a mixed fashion (down-up) or via their tops (up-up).

Since  $T_x$  goes down to infinity, a down-down connection is precluded, except of course if the leaf re-traverse the box  $B$  but then a foliated disc is created, by doubling the first return to the cross-section of  $B$  (Fig. 11.c). We use here Schoenflies (2.6) (or a simple form thereof) in the 1-connected Moore surface  $U \approx M$  (taking advantage of the canonical open cover by thorns aggregation). The same argument excludes the possibility of a down-up connection. Thus we have an “up-up” connection and then sides of the leaf  $L$  are squeezed between the three thorns  $T_{x'}, T_x, T_{x''}$  and this without the possibility of coming up again (else a foliated disc is created by the same Schoenflies mechanism). This triple squeezing clearly impedes the denseness of  $L$ , proving our assertion. ■

## 7.5 Experimental data: prescribing topology and foliated dynamics

Now we try to draw a picture showing the interplay between the topology and the possible foliated dynamics for surfaces. This involves a Venn diagram with the following topological versus foliated attributes:

- (1) Combinatorial topology: simply-connected  $\Rightarrow$  dichotomic;
- (2) Point-set topology: metric  $\Rightarrow$  separable;
- (3) Foliated dynamics: minimal  $\Rightarrow$  transitive  $\Rightarrow$  foliated.

Recall also some ‘transverse’ implications: transitive implies separable (7.3), and the mutual exclusion of 1-connected and transitive (5.1) or (6.4). By way of examples the following diagram (Figure 12) shows that this is a reasonably exhaustive list of obstructions. A closer look aids guessing new empirical obstructions, or more neutrally to ask the right questions. For instance it looks rather hard to exhibit a separable, 1-connected, non-metric surface lacking a foliation. Recall however that it is possible (yet not very easy) to locate separable non-metric surfaces lacking foliations (cf. the mixed Prüfer-Moore surfaces in BGG2 [5]).

Let us describe the various regions of the diagram (the following enumeration refers to the labels (1), (2), (3), etc. fixed on Figure 12 in a spiral-like fashion):

(1) and (2) contains respectively only the plane and the sphere which are the only simply-connected metric surfaces (2.5).

(3) is a region where it is difficult to exhibit a specimen. (For a vague candidate cf. Section 7.7)

(4) contains the long-glass  $\Lambda_{0,1}$  which is the long-cylinder  $\mathbb{S}^1 \times \mathbb{L}_{\geq 0}$  capped-off by a 2-disc. This surface cannot be foliated by BGG1 [4].

(5) has a plethora of examples including the long-plane  $\mathbb{L}^2$ , the long-quadrant  $\mathbb{L}_+^2$  and the original 1-connected Prüfer surface  $P_{\text{collar}}$ .

(6) contains the Moore surface  $M$  which has a foliation induced by the vertical foliation of the bordered Prüfer surface  $P$ . The latter has semi-saddle singularities disappearing during the folding process  $P \rightarrow M$ . Also in (6) we have the more exotic Maungakiekie surface, which is the result of a long Nyikos expansion effected to an open 2-cell.

(7) contains naively speaking  $2P$  the doubled Prüfer (alias Calabi-Rosenlicht manifold) which has a horizontal foliation. This does not preclude the possibility that  $2P$  endowed with a more exotic foliation is transitive. Thus here we ignore



has already been positioned in a deeper nest of the diagram.

(13) could contain the Moorization  $M(G)$  of a multiply-connected domain  $G = \Sigma_{0,n}$  with at least  $n \geq 3$  contours (starting with the “pant”). A vague idea could be that the Moorization forces a vertical behavior near the cytoplasmic expansions present in the Moorization, thus “shaving” the “hairs” gives a compact subregion (homeomorphic to  $G$ ) along the boundary of which the foliation is transverse, and then we are done by the Euler-Poincaré obstruction. But this hasty intuition is wrong as shown in Section 7.6. In particular this argument would imply that the Moorized disc cannot be foliated, but (7.8) below shows the contrary. The same construction (cf. Fig. 13g) also shows that the  $M(\Sigma_{0,n})$  for  $n \geq 3$  admit foliations, and so belongs to region (7), but not to (8) in view of (7.7), the case  $n = 3$  following also from (6.7). So region (13) looks rather deserted, except if we remind the construction in BGG2 [5, Sect. 4.2] of mixed Prüfer-Moore surfaces which produces a bunch of separable, non-simply-connected surfaces lacking foliations. Furthermore if we accept the operation of *full* Nyikosization  $N$ , which produces long hairs at all point of the boundary, then  $N(\Sigma_{0,n})$  for  $n \geq 2$  would belong to (13), compare Section 7.7.

(14) has the surfaces  $\Lambda_{0,n}$  of genus 0 with  $n \geq 3$  long cylinder-pipes, which lack foliations by BGG1 [4] (super-massive black hole scenario).

(15) admits a plethora of examples with most of them arising indeed from a non-singular flow (cf. [14] and recall optionally the theorem of Whitney [43] (building over Hausdorff) telling that non-singular 2D-flows induce foliations). Of our pictured examples the only one which is not induced by a flow is the “wormhole” double long-plane, i.e. the connected sum  $\mathbb{L}^2 \# \mathbb{L}^2$  which is however foliated by circles.

(16) contains simple examples using variants of the Prüfer construction. For instance we can Prüferize an annulus and glue radially opposite boundaries by long bridges (cf. figure).

(17) The same construction as in (16) with short bridges yields a specimen.

(18) We lack a serious example. However with the operation of full Nyikosization we can take  $N(\Sigma_{g,n})$  where the genus  $g \geq 1$  and with  $n \geq 1$  contours.

(19) Take a Kronecker torus Prüferized along an arc. By the scenario of nano-black holes (7.6) this surface is not minimal, giving a sharp positioning.

(20) Puncture the Kronecker torus, and making a long Nyikos expansion of one of the 2 separatrices converging to the puncture gives a minimal foliation on a non-metric surface (trick due to M. Baillif, more details in BGG2 [5]).

(21) Take a Kronecker torus. This is the unique compact specimen as follows from Kneser (6.1), but there is of course a menagerie of non-compact examples (e.g., Kronecker torus punctured once).

(22) Take the example of Dubois-Violette on  $S_{4*}^2 = \mathbb{R}_{3*}^2$ .

(23) contains as fake specimen the Kronecker torus punctured twice on the same leaf. (In reality the twice punctured torus also carries a minimal foliation, so it is not a sharp example.)

(24) Take the torus with the trivial foliation by circles. Of course this is fake, since the torus really lives in (21). Yet (24) contains the Möbius band (=twisted  $\mathbb{R}$ -bundle over  $S^1$ ) which is intransitive by (6.6). Doing one (or even 2) punctures in Möbius the surface is still intransitive by (6.8) resp. (6.16). For a compact example we have the Klein bottle  $\mathbb{K}$ , which by Kneser (6.1) is not minimal, and in fact foliated-intransitive (6.2).

(25) contains all closed surfaces except those having already been positioned, namely the sphere, and the 2 surfaces with Euler characteristic zero. These split into orientable surfaces of genus  $\geq 2$  and non-orientable surfaces (spheres with  $g \geq 1$  cross-caps) for all values of  $g$  except  $g = 2$  which is the Klein bottle. Since open metric surfaces always foliate (Morse function argument (4.10)), this is a complete tabulation of the birds in class (25) in view of the classification (2.4).

## 7.6 Razor principle foiled

Given the Moorized disc  $M(D^2)$ , which looks like a hairy disc with hairs emanating transverse to the boundary (Fig. 13a), one could expect that a foliated structure has to be compatible with the hairs, and thus ‘shaving’ the hairs gives an impossible foliation of the (compact) disc. This would imply that  $M(D^2)$  lacks a foliation. This naive principle is erroneous.

Basically, it is faulty because the *semi-saddle*  $xy$  (level curves of that function) restricted to  $y \geq 0$  is not the unique one inducing a regular foliation after Moorization. Indeed the *half-saddle* defined by  $x^2 - y^2$  (cf. Fig. 13c) behaves also well under folding. This suggests how to foliate  $M(D)$ , compare Figure 13, which is commented upon in the picture-assisted proof below.

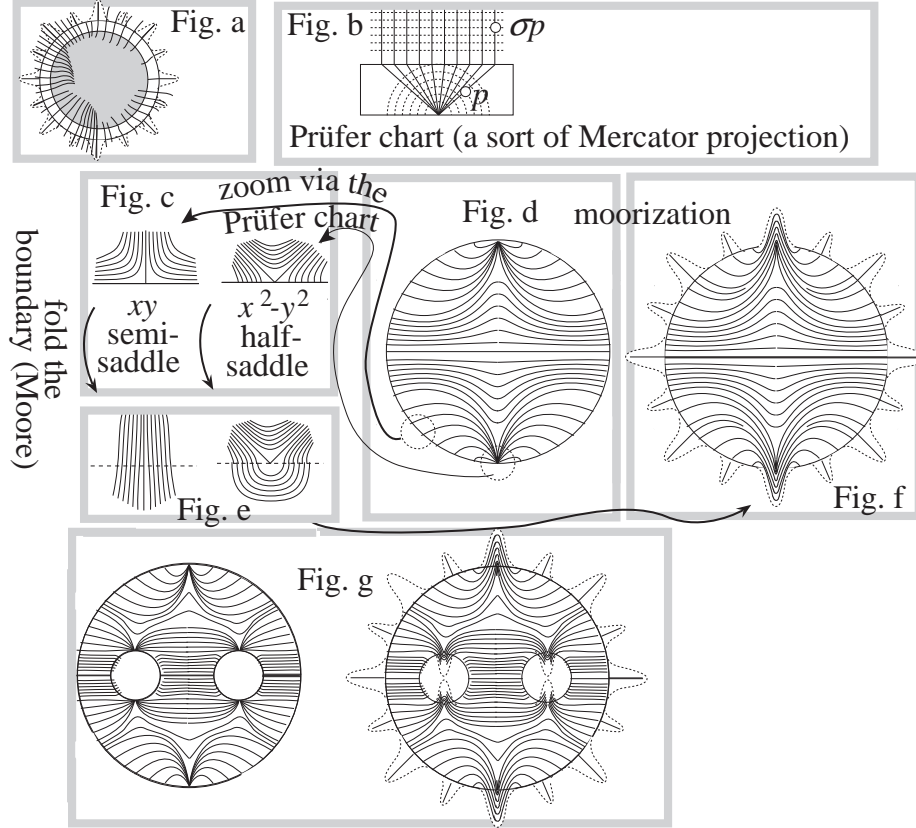


Figure 13: Foliating the Moorized disc

**Proposition 7.8** *The Moorized disc can be foliated.*

**Proof.** Consider the singular foliation of the disc  $D^2$  depicted on Fig. 13d: this is everywhere orthogonal to the boundary except at 2 singular points resembling a ‘spiderman’. Recall the interpretation of Prüfer’s construction in terms of rays, and the synthetic description of the (Prüfer)-charts. This can be thought of as the map  $\sigma$  pictured above (Fig. 13b). It is easy to see that if the foliation is vertical then its image by the Prüfer chart is a semi-saddle, whereas the spiderman singularity transforms to a half-saddle. Since both types of singularities disappear during the folding process (Fig. 13e), we get a genuine foliation on the Moorization  $M(D^2)$  (depicted on Fig. 13f). ■

The same method shows that the Moorization of the multiply-connected domains (any number of contours) can be foliated (Fig. 13g).

## 7.7 Razor principle for long hairs, supernovas and science fiction

It is rather hard to find non-metric separable surfaces lacking foliations. However, in [5, Sect. 4.2], Baillif showed that suitable mixed Prüfer-Moore surfaces (which are separable) lack foliations. The idea is the following. Start with  $P$  the bordered Prüfer surface. Each contour (=boundary component) of  $P$  can either be folded (Moorization) or left unaltered. Since Moorization is well behaved w.r.t. the vertical foliation (Fig. 13c-e), but not against the horizontal foliation developing thorn singularities when folded, Baillif showed that a violent mixture of both processes (folding and ‘nothing’) produces bordered surfaces whose double are separable, yet without foliations. Being doubled such surfaces fails to be 1-connected, and it would be interesting to answer the:

**Question 7.9** *Is there a simply-connected non-metric separable surface without foliations? If yes, then more subjectively, what is the simplest such example?*

In the second subjective question, “simplest” seems to have a double fragrance, namely simplest to construct or simplest to show its non-foliability. Along the second interpretation, we have perhaps the following (very hypothetical) answer, involving another black hole scenario—this time at the ‘mesoscopic’ scale! *Warning:* the next paragraph is maybe only pure science fiction.

Recall Nyikos’ long (cytoplasmic) expansion of a cell (cf. [5]). Assume such long expansions can be performed as often as we please, doing them in all directions of a disc yields a 1-connected separable surface with many long hairs emanating in all directions (naively imagined as orthogonal to the circumference). Call this manifold the *supernova* (with long hairs and complicated corona). Another more tangible generating mode for the supernova is to do first a Moorization (of the disc), and then make Nyikos’ expansions to the thorns (conceived now as independent processes). It is conceivable that any foliation of the supernova contains all hairs as semi-leaves, by a variant of (7.6). (I think that this was verified in David Gauld’s hand-written notes from the parc Bertrand, Geneva 2010, unfortunately unpublished as yet.) Then one might (via a razor principle) deduce a foliation of the compact disc (transverse to the boundary). The supernova would thereby answer positively our naive question (7.9). Albeit it is not the simplest to construct, it is perhaps the easiest to show its lack of foliated structure.

## 8 Miscellaneous

This section collects miscellaneous topics not directly relevant to the main text:

### 8.1 Jordan separation

Maybe first a question related to Jordan separation approached via the Riemann polarization trick. Recall that we defined a divisor in a manifold as a locally-flat hypersurface which is closed as a point-set (5.4). Taking for granted the universality (i.e., non-metrical validity) of the Riemann trick (5.5) we deduced that *in a simply-connected manifold any divisor is dividing* (5.7). The converse does not hold, for instance the Poincaré (homology) sphere is divided by any divisor, but not simply-connected. The former assertion follows from the well-known:

**Lemma 8.1** *Given a closed (topological) manifold  $M$  whose first homology modulo 2,  $H_1(M, \mathbb{Z}_2)$  is trivial (equivalently the first Betti number  $\beta_1$  with mod 2 coefficients, is zero). Assume furthermore that the manifold as a reasonable “intersection theory”, which is certainly the case whenever  $M$  is smoothable (Weyl 1923, Lefschetz 1930, de Rham 1931, etc.). Then  $M$  is divided by any divisor.*



**Proof.** By contradiction, let  $H$  be a non-dividing divisor. Since  $H$  is closed as a point-set in  $M$  compact, it is compact. Thus it carries a fundamental class mod 2, denoted  $[H]$ . Choose any point  $p \in H$  and a locally flat chart  $U$  with  $(U, U \cap H, p) \approx (\mathbb{R}^n, \mathbb{R}^{n-1} \times \{0\}, 0)$ . Fix a little arc  $a$  transverse to  $H$  and meeting it in exactly one point. Since  $M - H$  is connected, choose a path  $b$  in  $M - H$  joining both extremities of  $a$ . Thus  $a + b =: c$  defines a 1-cycle (mod 2) intersecting  $H$  in exactly one point transversally. Thus the intersection number  $[c] \cdot [H] = 1$ . This implies that  $[c] \neq 0$ , and so  $\beta_1 \neq 0$ . ■

Consequently the Poincaré sphere,  $M^3$ , splits because  $H_1(M^3, \mathbb{Z}) = 0$  implies  $H_1(M^3, \mathbb{Z}_2) = 0$  (true in any space, for a 1-cycle mod 2 can always be oriented). We do not know if the lemma holds in full generality:

**Question 8.2** *Assume the manifold  $M$  to have  $\beta_1 = 0$ , then is  $M$  separated by any divisor? And what about the converse?*

## 8.2 Grötzsch-Teichmüller theory for non-metric surfaces?

Since Radó 1925 [37], it is known that a complex-analytic structure on a 2-manifold implies second countability, hence metrizability (Urysohn). Now such a structure also known as a Riemann surface structure is essentially the same as a conformal structure allowing one to measure angles. In fact this can be achieved in the more general category di-analytic or Klein surfaces, where non-orientable surfaces are permitted. Radó's theorem generalizes directly to Klein surfaces, by passing to the orientation double cover which is a Riemann surface (with an anti-holomorphic involution):

**Lemma 8.3** *Any Klein surface is metric.*

**Proof.** Above the Klein surface there is a Riemann surface, which by Radó [37] is metric, thus Lindelöf and the latter property pushes down to the Klein surface, which being locally second countable is then second countable, and Urysohn concludes. ■

Since conformal structures are lacking on non-metric surfaces, and reminding the quasi-conformal trend (initiated in the now classical works of Grötzsch, 1928, Lavrentief 1929, Ahlfors 1935, Teichmüller 1938, etc.) it is rather natural to wonder about quasi-conformal structures. This was addressed by R. J. Cannon 1969 [9], who found rather surprising answer(s). More on this soon, yet let us first dream a little.

If a diffeomorphism (between regions) of the plane (say of class  $(C^1)$ ) is not conformal, then it will distort infinitesimal circle into ellipses, whose eccentricity  $Q \geq 1$  (long axes divided by the short axes) provides a dilation quotient measuring the deviation from conformality. The diffeomorphism is *K-quasiconformal* if the distortion  $Q$  is bounded by a finite constant  $1 \leq K < \infty$  throughout its domain of definition.

A *K-quasiconformal structure* is defined by an atlas with transition maps being *K-qc*. Of course a 1-qc structure is nothing else than a Klein structure, or a conformal structure (non-orientable surfaces are welcome, at least permitted).

Now the dream would be that given any (non-metric) surface (say with a differentiable structure, albeit there is a way to speak of quasi-conformality without regularity assumption), then (following Grötzsch's idea and phraseology) we could look for the “*möglichst Konform*” atlas minimizing the angular distortion  $Q$ . Thus there is a way of assigning to every surface  $M$  a number  $1 \leq Q(M) \leq +\infty$  which is the infimum of  $K$  over all *K-qc* atlases. Of course we set  $Q(M) = +\infty$  if there is no such atlas, and  $Q(M) = 1$  holds precisely when  $M$  is metric (Radó's theorem). This provides a continuous numerical invariant quantifying how violently non-metric the surface is. Unfortunately it seems that there is no such fine quantification, for Cannon's main result is that any *K-quasi-conformal* structure on a surface forces metrizability so that  $Q(M)$  can take only the values 1 or  $+\infty$ .

However Cannon also shows that some reasonable surfaces (of the Prüfer type) as well as surfaces deduced from the long ray (Cantor type) allows quasi-conformal structures in the weak sense that there is no uniform bound on the distortion valid for all transition maps. This raises the question if such (weak) quasi-conformal structures exist for all surfaces. (Maybe this would follow from the conjectural smoothability of surfaces.)

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